Part I.

Theoretical Basis

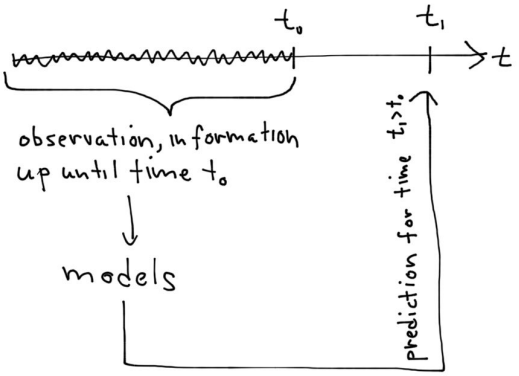
Contents

1. Dynamical Systems

1.1. General concepts

Since most complex systems involve dynamics, the change of things over time, e.g. in growth processes in pattern formation in collective motion and behavior, we need to disscuss some general ideas on how to approach these phenomena mathematically or quantitatively.

A model dynamical systems is the basic foundation of predictive science. That means: if we know something about the past, we ca say something about the future. Sometimes these predictions are probabalistic, e.g. we can only say how the probability of something occuring evolves over time or how large it will be at some point in the future. Before we discuss various forms of models, let's introduce some very general concepts.



- **State:** $\rho(t)$ This is a quantity that encapsulates the information on a natural system at some time t
- **History:** $G_{t_0} = \{\rho(t) : t \leq t_0\}$ This is the collection of states of the system for past times until time t_0
- **Dynamics:** $\rho(t_n) = D_{t_n,t_0}(G_{t_0})$, $t_n > t_0$ maps history onto future stystem states. This can be a very complicated function. It takes as an argument the history G_{t_0} and can depend explicity on the time of prediction t_0 and the future time t_n
- **Time evolution:** In most dynamical systems time either evolves

– continuously $-\infty < t < \infty$or

– discretely $t_n \quad n \in \mathbb{Z}$ (stroboscopic information)

1.1.1. Typical state variable $\rho(t)$

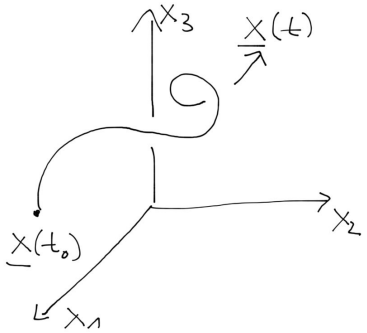
In most situations the state ρ takes the form of an n-component vector of real numbers, so for continuous time

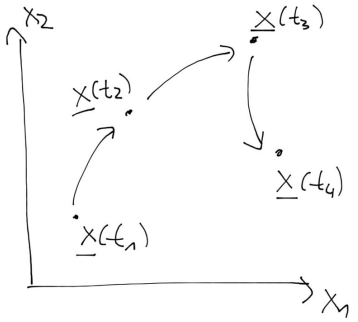
$$\rho(t) = \mathbf{x}(t) = (x_1(t), ...x_m(t))$$

and for discrete time

$$\rho(t_n) = \mathbf{x}(t_n) = (x_1(t_n), ...x_m(t_n))$$

This allows us to visualize the dynamics as a trajectory in **phase space** which is the vector space spanned by the components of the state vectors.





Sometime, also in this course, the state is given by an entire function, called a field $u(\mathbf{x}, t)$ where $\mathbf{x}$ now refers to a position in physical space for example and $u(\mathbf{x}, t)$ is the value of the field at position $\mathbf{x}$ and time t. Sometimes we are dealing with multiple fields at the same time, each one we can think of as the component of a vector field

$$\mathbf{u}(\mathbf{x}, t) = (u_1(\mathbf{x}, t), ..., u_m(\mathbf{x}, t))$$

For example the state of the system could be the concentration $u(\mathbf{x}, t)$ of a substance, the probability of a particle being at some location $\mathbf{x}$ at some time t etc. Also in this case we can have discrete time an continuous time, or discrete space or continuous space or combinations of both. We will see examples of all combinations throughout this course. Additionally, the positional variable $\mathbf{x}$ can be generalized to refer to non-metric coordinates, for example it could label the nodes on a complex network and $\mathbf{u}(\mathbf{x}, t)$ could refer to the values of the vector field at node $\mathbf{x}$ at time t.

1.1.2. Markovian dynamics

Very many natural systems evolve according to a dynamics that "forgets" is history and the new state only depends on the most recent state and not on the entire history. These systems are called Markovian. In other words the dynamics $D_{t,t_0}(G_{t_0})$ only depends on $\rho(t_0)$ and not on G_{t_0} (remember $G_{t_0} = \{\rho(t); t < t_0\}$)

$$\rho(t_n) = D_{t_n,t_0}(G_{t_0}) = D_{t_n,t_0}(\rho(t_0))$$

Therefore, in this case the dynamics D maps states onto states.

1.1.2.1. Discrete time

In discrete time this is straighforward to understand and the dynamics can be written as

$$\mathbf{x}(t_{n+1}) = \mathbf{F}_n(x(t_n)) \qquad \mathbf{x}(t_n) \in \mathbb{R}^n \tag{1.1}$$

Note that the dynamics takes the form of a vector function $\mathbf{F}_n$ and that this can explicitly depend on the time index n, meaning that it can change at every time step.

1.1.2.2. Continuous time

In continuous time the above equation takes the form

$$\mathbf{x}(t) = \Phi_{t,t_0}(\mathbf{x}(t_0))$$

and the dynamics Φ is called a flow and it is defined for $t \geq t_0$ and has the peculiar property of being the identity when $t = t_0$. However, in real applications, one ususally never discusses these flows. Rather one defines the dynamical systems by specifying the short time evolution of a state. We say that Δt is a small time interval and

$$\mathbf{x}(t + \Delta t) = \mathbf{x}(t) + \Delta t \times \mathbf{f}(\mathbf{x}(t), t) + \mathcal{O}(\Delta t^2)$$

which amounts to

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(x(t), t) \tag{1.2}$$

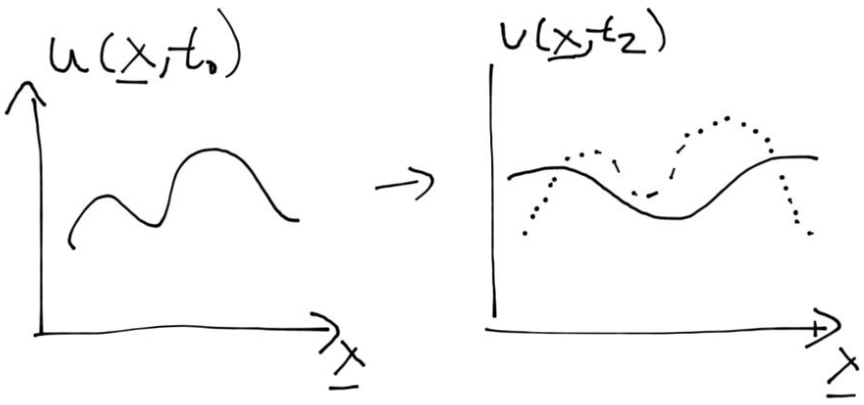
which is an m-dimensional first order system of ordinary differential equations.

1.1.2.3. Field equations

If a system's state is defined by a field $\mathbf{u}(\mathbf{x}, t)$ the generalization of the above becomes a field equation

$$\partial_t \mathbf{u}(\mathbf{x}, t) = f[\mathbf{u}(\mathbf{x}, t), t]$$

where f is called a functional which takes as an argument the vector field $\mathbf{u}(\mathbf{x}, t)$. In this case f tells us what the entire vector field $\mathbf{u}(\mathbf{x}, t)$ is going to be, given the function at an earlier time t_0.



1.1.3. Autonomous dynamics

Autonomous dynamics is a special case in which the functions that map states on new states do not explicitely depend on time. The system evolves "by itself" without external input, in discrete dynamical systems this reads

$$\mathbf{x}(t_{n+1}) = \mathbf{F}(\mathbf{x}(t_n))$$

and

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t))$$

1.1.4. Stochastic and deterministic dynamics

When a system evolves stochastically the functions $\mathbf{F}_n$ and $\mathbf{f}(\cdot, t)$ are known only in a statistical sense, i.e. one only knows (at best) the probability with which a mapping occurs from an ensemble of possible mappings. If $\mathbf{F}_n$ and $\mathbf{f}(\cdot, t)$ are known with certainty, the dynamics is called deterministic. We will later see that also deterministic dynamics can behave like stochastic systems.

1.1.5. Rules and function

Sometimes a dynamical system, say of discrete type

$$\mathbf{x}(t_{n+1}) = \mathbf{F}(\mathbf{x}(t_n))$$

can be expressed using a mathematical function $\mathbf{F}$, for example,

$$x(t_{n+1}) = \sin(x(t_n) + 7).$$

Other times, the "function" $\mathbf{F}$ is better expressed as a set of rules. For example if the state variable is a sequence of 1's and 0's say

$$\mathbf{x}(t_n) = 0110101001101$$

the rule $\mathbf{F}$ could be:

1. Replace all pairs of 1's with the sequence 10
2. Then replace all triplets 101 with 111
3. Then replace all 00 pairs with 01

1.1.6. Nonlinear Dynamics

The reason why systems governed by

$$\mathbf{x}(t_{n+1}) = \mathbf{F}(\mathbf{x}(t_n))$$

or

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t))$$

are interesting and challenging is because in general the function $\mathbf{F}$ and $\mathbf{f}$ are nonlinear and cannot be solved. For example, later we will have a close look at

$$x(t_{n+1}) = 4x(t_n)\left[1 - x(t_n)\right]$$

which is a very simple quadratic equation but which bears a lot of dynamical surprises.

1.1.7. Notation

In this course we will make use of notations which have become a convention in most dynamical systems research. Instead of

$$\frac{d\mathbf{x}}{dt} \quad \text{or} \quad \frac{d^2\mathbf{x}}{dt^2}$$

we will frequently use the notation:

$$\dot{\mathbf{x}}(t) = \frac{d\mathbf{x}}{dt} \quad \text{and} \quad \ddot{\mathbf{x}}(t) = \frac{d^2\mathbf{x}}{dt^2}$$

For partial derivatives we use the notation

$$\partial_{x_i}\mathbf{f}(\mathbf{x}) = \frac{\partial}{\partial x_i}\mathbf{f}(\mathbf{x})$$

Also instead of writing

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), t)$$

we will write simply

$$\dot{\mathbf{x}} = f(\mathbf{x}, t)$$

keeping in mind that a trajectory $\mathbf{x}(t)$, a solution to the above dynamical system, is a function of time.

1.1.8. Higher Order Systems

Let's look at continuous time dynamical systems as defined above

$$\dot{\mathbf{x}} = f(\mathbf{x})$$

and for simplicity consider an autonomous system. It may appear that a restrictions of this type would exclude dynamical systems of higher order, e.g.

$$\ddot{\mathbf{x}} = f(\dot{\mathbf{x}}, \mathbf{x})$$

and example are systems governed by Newtonian mechanics, e.g.

$$m\ddot{\mathbf{x}} = F(\dot{\mathbf{x}}, \mathbf{x})$$

where F is a function that's acting on an object of mass m. The restriction to first order is plausible though because any higher order system can be mapped on a first order system in a phase space of higher dimension. For example we can write

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{p} \\ \dot{\mathbf{p}} &= F(\mathbf{p}, \mathbf{x}) \end{aligned}$$

and the state of the system at time t is given by the pair of vectors $(\mathbf{x}(t), \mathbf{p}(t))$ which has twice the dimension of the $\mathbf{x}(t)$.

1.1.9. Asymptotic behavior

Given a dynamical system one is generally interested in the asymptotic behavior, the long time limit, $t \to \infty$. Generally this behavior comes in different flavors.

1.1.9.1. Fixed points

Usually one first tries to find states that do not change under the dynamics. These are called stationary solutions, fixed points or equilibria. These are points that fullfill

$$f(\mathbf{x}^\star) = 0$$

for flows, because that implies that $\dot{\mathbf{x}} = 1$. And for discrete dynamics it implies

$$F(\mathbf{x}^\star) = \mathbf{x}^\star$$

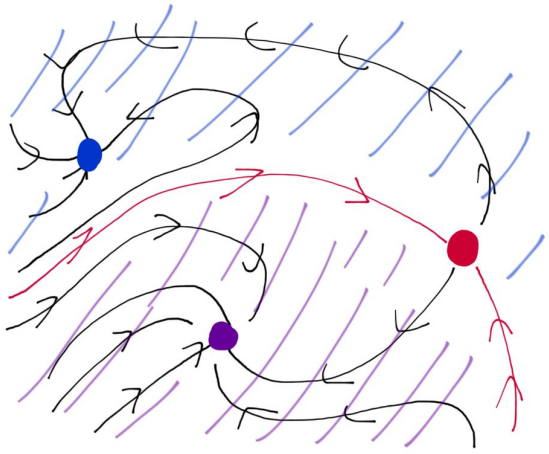
because that also implies in these systems that nothing changes. Fixed points come in two categories: stable and unstable. Stable means that if we perturb the system away from the fixed points, the system will go back to it. Unstable means that any arbitrarily small pertubation will increase and the state will diverge away from it. There are, as always more exotic situations and definitions of stability but we won't need those here. There are covered in courses on dynamical systems theory.

1.1.9.2. Fixed points as attractors

The system can approach a stationary state, it can happen that

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^\star$$

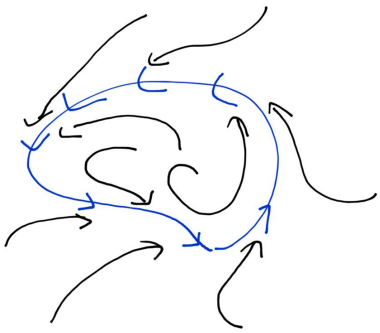
for some initial condition. Of course a system can have multiple such fixed points. When they attract a bunch of initial conditions they are called attractors and the set of points they attract are called basins of attraction. The way that trajectories approach such an attractor can be different, we will discuss various types of attraction later when we discuss 2D flows. Here is an example of three attractors that are fixed points:



The black lines indicate tranjectories of the dynamical system. The blue and the purple stationary or fixed point attract all the trajectories in the area with the same shade. The red fixed point is a bit different. It only attracts points along the red lines. This fixed point is unstable, because any arbitrarily small perturbation of a trajectory along the red line will end up in either the blue or purple attractor. The red line is also called a separatrix.

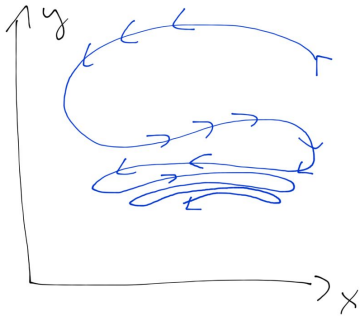
1.1.9.3. Limit Cycles

Before we understand limit cycles, we have to know the concept or periodic orbit. This is a solution to a dynamical system that comes back to a starting point and since we consider deterministic systems, the evolution will repeat itself. If such a periodic solution attracts trajectories from elsewhere it's called a limit cycle. The concept is very straighforward, geometrically it looks like this:



1.1.9.4. The Poincare Bendixon Theorem

One of the most fundamental things concerning flows is that 1D flows cannot have periodic orbits, only fixed points and that in 2D the Poincare Bendixon Theorem holds. It essential states that if we have a 2d system, asymptotically the system can only approach either fixed points (including ∞) or limit cycles. This is very deep stuff, and we won't go into it here. But it's important to remember that asymptotically nothing more fancy is to be expected of 2d time continuous flows. You may think that this is obvious but it's not. For example something like this:



can actually not occur in 2D which is really weird. This is all only true for flows, not for maps. In fact, for instance the simple 1d map

$$x_{n+1} = -x_n$$

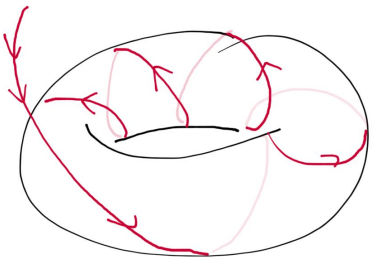
exhibits an infinite number of periodic orbits and the simple map

$$x_{n+1} = 4x_n(1 - x_n)$$

exhibits motion with asymptotically neither approaches a fixed point nor a limit cylce.

1.1.9.5. Quasi-Periodicity and Tori

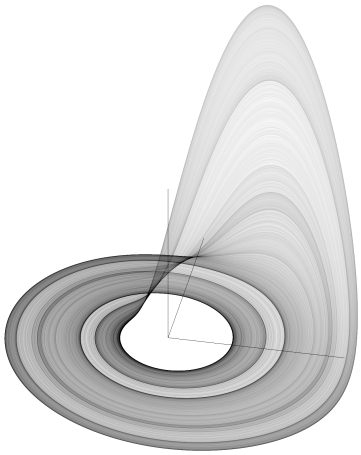
If our system evolves in high dimension, we can have generalizations of the limit cycle as an attractor, e.g. a torus which can be thought of a two limit cycles multiplied with one another:



This torus can attract a bunch of trajectories, the interesting thing is that trajectories on a torus no longer have to be periodic. But almost. That's why they are often called quasi-periodic.

1.1.9.6. Strange attractors

One might think that the asyptotic behavior of a dynamical system $\dot{\mathbf{x}} = f(\mathbf{x})$ could only be one of the above. The interesting thing is that some systems also possess strange attractors which do not fall into the above categories. These are very interesting and we will discuss them later when we discuss chaotic systems. Here's a picture of a trajectory of a system that has a strange attractor:



It looks very simple at first sight, but if you look at it closely you see something is really *strange* about it. It looks two dimensional but there are two sheets that fold into one another and in fact trajectories that approach this attractor are unpredictable in the sense that one cannot say when a trajectory will take a loop up and when not. The dynamical system that generates this are actually quite simple in form:

$$\begin{aligned} \dot{x} &= -y - z \\ \dot{y} &= x - Ay \\ \dot{z} &= B + z(x - C) \end{aligned}$$

1.1.10. Asymptotic behavior of time discrete maps

Essentially the situation is similar when we look at discrete maps (where time is denoted by the index n).

$$\mathbf{x}_{n+1} = f(\mathbf{x}_n).$$

Here a stationary solution is one for which

$$\mathbf{x}_n = f(\mathbf{x}_n)$$

and a limit cycle with period M is one for which

$$\mathbf{x}_{n+M} = \underbrace{f \circ f \circ .... \circ f}_{M}(x_n).$$

This means that we apply the function M times and end up at the starting point again. We will see, that even the simplest maps can non-periodic behavior that does not approach a fixed point. Very soon.

1.1.11. Bifurcations

In addition to the asymptotics of a dynamical system, we are often also interested in how this asymptotic behavior changes with parameters of a system. For example if we have a system

$$\dot{\mathbf{x}} = f(\mathbf{x}; \mathcal{P})$$

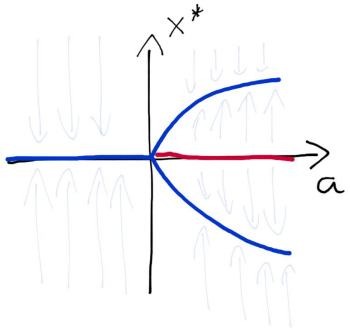
it typically depends on a set of parameters

$$\mathcal{P} = \{p_1, ..., p_k\}.$$

And for different sets of these parameters the system may behave differently. It is then important to determine what parameter choices yield what type of asymptotic behavior. In this, it's important to determine boundaries in parameter space where this behavior changes. These are often called bifurcations, for reasons that will become clear soon. For example, let's assume we have a dynamical system

$$\dot{x} = ax - x^3$$

where a is a parameter. For $a \leq 0$ this system has one attractive fixed point $x^\star = 0$, every initial condition will move towards this fixed point. If, however, $a > 0$ then the system will have three fixed points, the central one $x^\star = 0$ and two additional ones $x^\star_{1/2} = \pm\sqrt{a}$. It turns out that in this situation any initial condition $x(0) > 0$ will approach $\sqrt{a}$ and any initial condition $x(0) < 0$ will approach $-\sqrt{a}$. We can illustrate this in a simple bifurcation diagram, in which we illustrate "behavior" as a function of the parameter a:



1.2. Examples of dynamical systems

Let's now list a few examples of dynamical systems that fall into the classes discussed above. All of them are Markovian, some are autonomous, some evolve in discrete time, some in continuous time, in some the state is described by a field, in some by simple low-dimensional vectors. Some of them will be discussed in detail later, some are boring, some hide exciting complex behavior.

1.2.1. The Pearl-Verhulst system

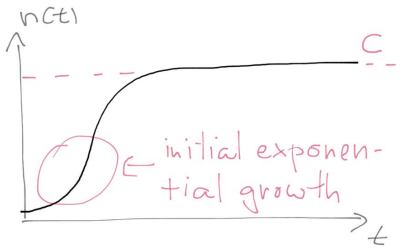
This is the simplest model for the growth of a population of animals or plants in an ecosystem, but it also has other applications. The state of the system is the density $n(t)$ of a species. And the dynamical system is given by

$$\dot{n} = \gamma(1 - n/c)\, n$$

This DS (dynamical system) is a one-dimensional, autonomous continuous time system with two parameters γ and c We see immediately that if $n(0) = 0$ or $n(0) = c$ then $\dot{n} = 0$, which is why these two values are called stationary solutions or fixed points. However, the first is unstable, and the other is stable, if for instance $n(0) = \epsilon > 0$ then

$$\dot{n} \approx \gamma n$$

which is positive and yields an initial exponetial growth. if $n(t)$ approaches c this growth is slowed down more and more and the system reaches its capacity c. One can solve this, but one doesn't have to in order to understand what's going on.

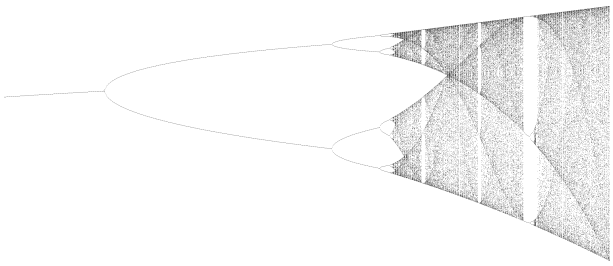


1.2.2. The logistic map

Similar to the Pearl-Verhulst system is the analogous discrete time version, called the logistic map

$$x_n = \lambda x_n(1 - x_n).$$

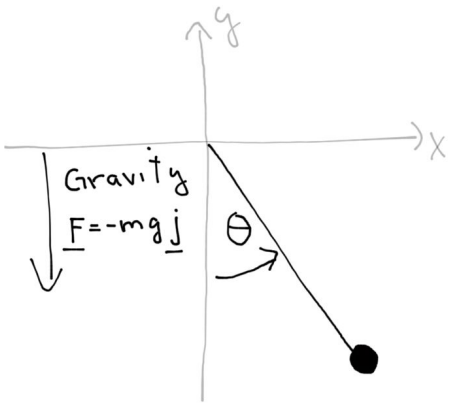
This is a one-dimensional autonomous map. Unlike the Pearl-Verhulst system it is actually a complex system, that can generate behavior much more complex that one would think just by looking at it. We will discuss this system in more detail soon. For instance, the following picture was generated with the logistic map:



We will not discuss what this figure shows now. It's only an appertizer for later.

1.2.3. The simple pendulum

Here's a simple mechanical system, the simple pendulum subject to gravity. A mass m is attached to a stick of length L which freely rotates without friction around a pivot at the point $(x, y) = \mathbf{0}$. The graviational force $\mathbf{F} = -mg\mathbf{j}$ acts on the mass ($\mathbf{j}$ is the unit vector along the y-axis). Although this system is so simple and certainly not a complex system (in fact it can only do very simple types of motion) its oscillatory potential represents something at the core of many complex dynamical phenomena. In other words, if you add a littlebit more to this system, it will become quite interesting.



The equations of motion are derive from Newtonian mechanics

$$m\ddot{\mathbf{x}} = \mathbf{F}$$

and the restrictions that

$$x = L\sin\theta \quad \text{and} \quad y = -L\cos\theta$$

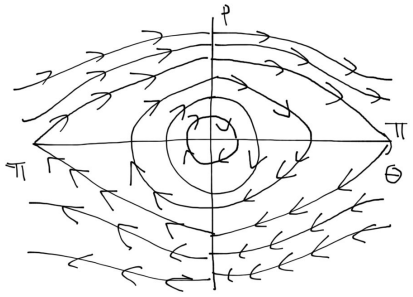
With a little bit of geometry one gets a second order ODE for the angular accelaration:

$$\ddot{\theta} = \frac{g}{L}\sin\theta$$

We can rewrite this as

$$\dot{\theta} = \frac{p}{mL^2} \quad \text{and} \quad \dot{p} = -mgL\sin\theta$$

which is a first order, two-dimensional dynamical system, the systems state is given by the pair (θ, p) and the trajectories evolve in the plane spanned by these two coordinates. This particular example has the special feature, that one variable, θ, lives on the circle $S = (-\pi, \pi]$ and not the ordinary infinite line. Although this system can be solved analytically, again, one doesn't have to do it to understand what's happening. In the 2D phase space trajectories look like this:



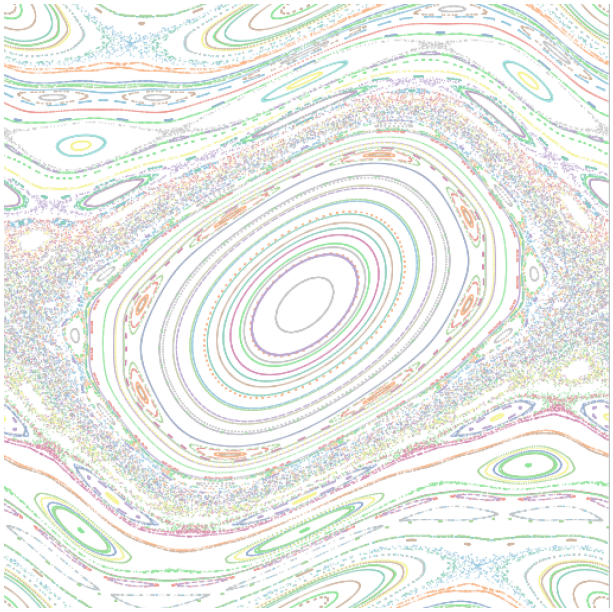
For small amplitudes the trajectories are circles, corresponding to small amplitude swinging motion. If p is large enough, the pendulum "flips" and keeps on rotating in one direction.

1.2.4. The kicked rotator

Here's a system related to the pendulum, but again a discrete version, this time a 2D map.

$$\begin{aligned} p_{n+1} &= p_n + K\cos(x_n) \mod 2\pi \\ x_{n+1} &= x_n + p_{n+1} \mod 2\pi \end{aligned}$$

We will not discuss here, how this equation, called the **standard map** arises, we will discuss this later in detail. For now, know that it is somewhat related to the pendulum in which instead of the continuous action of gravity on the pendulums mass, periodic kicks of strength K are transfered to the pendulum. Interestingly, this simple two dimensional system is capable of generating complex behavior, as we see here:



This picture depicts the sequence of points generated by the above map in the x, p plane for a bunch of random initial conditions. We will discuss the structure of this in more depth later.

1.2.5. The Fisher equation

Here's an example where the state of the system is not simply given by a vector of numbers $\mathbf{x}(t)$ but rather a whole function $u(\mathbf{x}, t)$ that is defined across space. Space could be 1,2,3 or n-dimensional space. Let's consider 1d. This equation

$$\partial_t u = \lambda u(1-u) + D\partial_x^2 u$$

where $u = u(x,t)$ is very famous. It combines the Pearl Verhulst system (the first term) for a concentration $u(x,t)$ with diffusion (the second term). It's a reaction-diffusion

system. The diffusion term implies that the concentration flows from high to low. So for example with we only have diffusion ($\lambda = 0$):

$$\partial_t u(x,t) = D\partial_x^2 u(x,t)$$

and we consider a discretization of space $x = n\Delta x$ then

$$u(x,t+\Delta t) \approx u(x,t) + \frac{\Delta t}{(\Delta x)^2} \times D \times [u(x+\Delta x,t) + u(x-\Delta x,t) - 2u(x,t)]$$

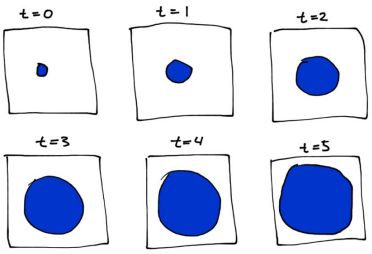
this means that the concentration around x increase by particles coming in from the right, $u(x+\Delta x,t)$ from the left $u(x-\Delta x,t)$ and leaving into both directions $2u(x,t)$. More about this later. In higher dimensions the Fisher equation reads

$$\partial_t u = \lambda u(1-u) + D\nabla^2 u$$

where

$$\nabla^2 = \left(\partial_x^2 + \partial_y^2 + ...\right)$$

The interesting feature of this equation is that it can generate waves that propagate at a constant speed:



Although the potential of generating propagating wave is interesting and not directly visible in the equation, it does not qualify as complex behavior. However, just like before, it carries the potential of generating complex behavior, as we see in the nect example

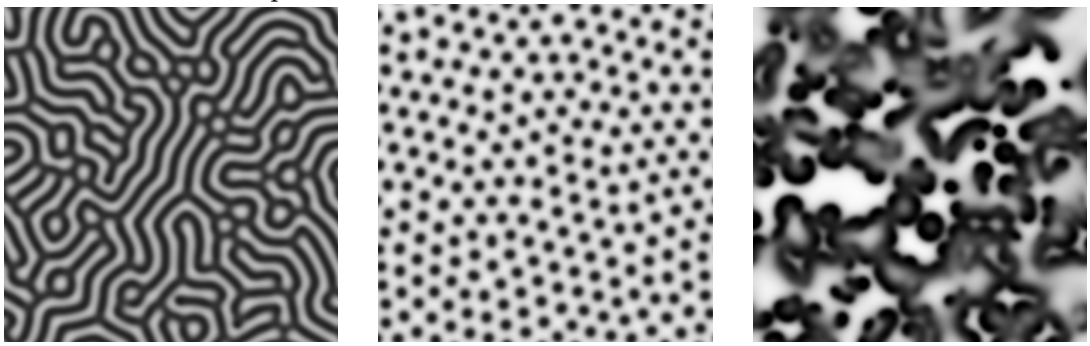
1.2.6. The Gray-Scott system

The Gray Scott Model is structurally very similar to the above example. It's a two dimensional reaction diffusion system:

$$\begin{aligned} \partial_t u &= -uv^2 + f(1-u) + D_u\partial_x^2 u \\ \partial_t v &= uv^2 - (f+k)v + D_v\partial_x^2 v \end{aligned}$$

where the dynamic state variables are the fields $u = u(\mathbf{x},t)$ and $v = v(\mathbf{x},t)$, usually 2d systems are considered, and f, k are two positive parameters and D_u, D_v are the

diffusion constant for each state variable. Later, we will discuss the motivation behind these equations in very much detail. For now, we just want to understand that these equations aren't much more complicated than the Fisher equation. However, they can generate and whole zoo of different behaviors, in this case a whole zoo of different patterns of u and v. Some examples:



1.2.7. Lindenmayer systems

Here's another type of dynamical system, which is very different in structure and more computer sciencish although the applications are predominantly in biology and mathematics. In fact Aristid Lindenmayer was a biologist and used these systems to describe the bodyplans of plants. A Lindenmayer System, also just called L-system essentially consists of variables, constants and rules and an initial condition (sometimes called axiom). The state of the system is typically a string σ_n of arbitrary size. A simple Lindenmayer system is this one:

$$\begin{aligned} A &\rightarrow ABA \\ B &\rightarrow BBB. \end{aligned}$$

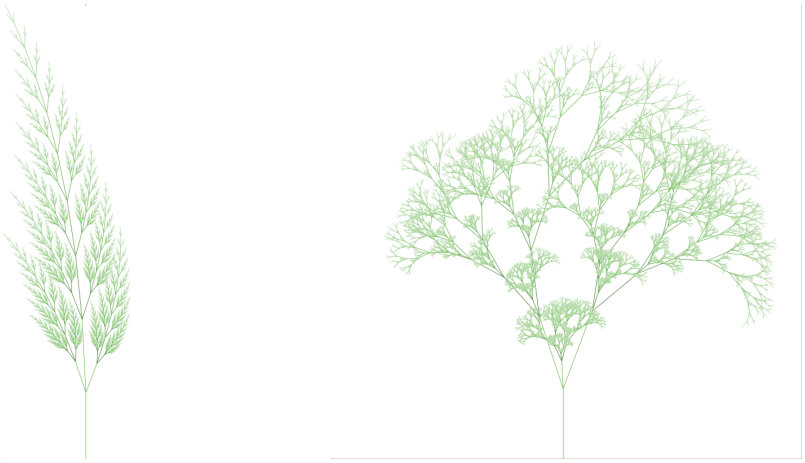
Given an initial string σ_0 for example

$$\sigma_0 = A$$

we apply rules iteratively and evolve the system this way:

$$\begin{aligned} \sigma_1 &= ABA \\ \sigma_2 &= ABABBBABA \\ \sigma_3 &= ABABBBABABBBBBBBBBABABBBABA \\ \sigma_4 &= ... \end{aligned}$$

We can see, that the states' representation becomes very long very quickly. This seems like a game, but it becomes interesting, when we give an interpretation to the symbols. With this type of dynamical system we can generate interesting shapes like these two:



1.2.8. The Viszek model

Finally, here's another interesting example of a dynamical system. This one is a many particle system, or system of agents. Every agent $i = 1, ..., N$ has a position $\mathbf{x}_i(t)$ a heading angle $\theta_i(t)$. Every particle moves at the same speed v. Assuming that everything is taking place in the plane, the dynamic state of the system say $\mathbf{X}(t)$ is therefore given by $3N$ numbers

$$\mathbf{X}(t) = (\mathbf{x}_1, \theta_1, ..., \mathbf{x}_N, \theta_N)$$

Now we let these particles move and interact. First, the move into the direction of their heading

$$\mathbf{x}_i(t+1) = \mathbf{x}_i(t) + \mathbf{v}_i(t)$$

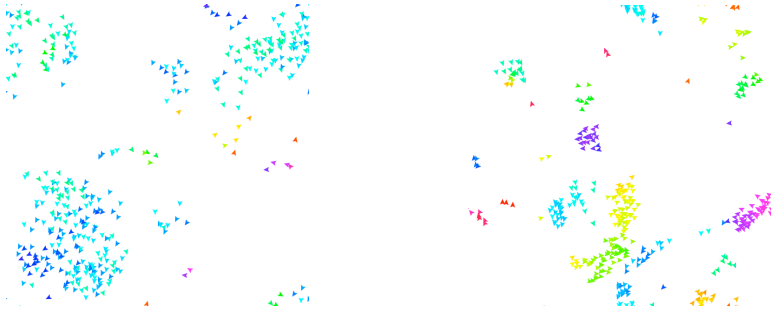
where

$$\mathbf{v}_i = v \begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix}$$

They interact by changing their direction according to the average $\langle\theta(t)\rangle_r$ heading of their neighbors (particles within a radius r) plus some random change $\sigma\Delta\theta$.

$$\theta_i(t+1) = \langle\theta(t)\rangle_r + \sigma\Delta\theta$$

where $\Delta\theta$ is a uniform random number between 0 and 2π and σ a constant parameter. This system generates different types of behavior, some of which is complex, e.g. it exhibits the ability for the individual particles to perform collective motion into one direction. We will study this type of system in detail later, here are two states of the system for different parameters:



1.3. One-dimensional flows

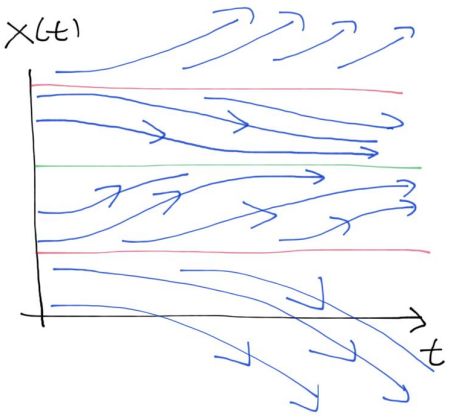
Let's now investigate the class of dynamical systems in which the state is just the number x and the dynamics are given by

$$\dot{x} = f(x)$$

What can we expect from simple systems of this type?

1.3.1. Absence of Oscillations

First let's ask if $\dot{x} = f(x)$ can have oscillations. Answer: No, because for chosen $x_1 = x(t_1)$ and $x_2 = x(t_2)$ with $x_1 = x_2$, we must have $f(x_1) = f(x_2)$, but for oscillations to occur, there must exist points where $x_1 = x_2$ where $\dot{x}_1 \neq \dot{x}_2$, but this is forbidden. Therefore, for regions/intervals of x, $x(t)$ must be monotonically increasing or decreasing or remain fixed.



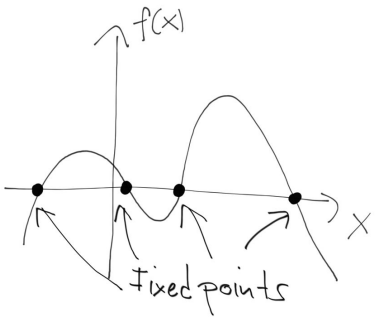
This raises the question about what happens at the points that separate these regions.

1.3.2. Fixed Points

Fixed points are the solutions of the equation

$$\dot{x} = f(x) = 0$$

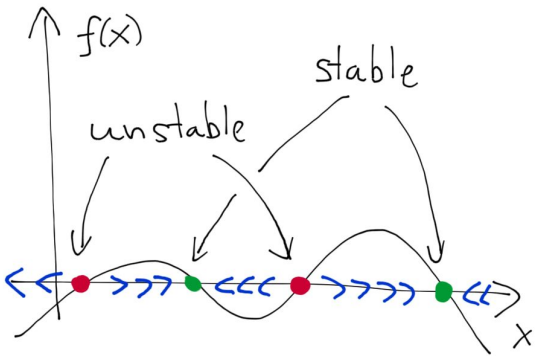
Of course a given system can have none, one, a few, many or an infinite number of fixed points. Those can be either computed analytically or determined by drawing $f(x)$ and observing when it hits the x-axis.



For example $\dot{x} = x^4 + 1$ has no fixed points, $\dot{x} = -x - x^2$ has two fixed points: $x^* = -1, 0$, $\dot{x} = \sin(x)$ has an infinite number of fixed points $x^* = n\pi$, $n = 0, \pm 1, \pm 2, ...$

1.3.3. Stability of Fixed Points

Stability of fixed points can be determined from the graphics as well by noting that $\dot{x} > 0 \Longrightarrow f(x) > 0$ means $x(t)$ increases, and $\dot{x} < 0 \Longrightarrow f(x) < 0$ means $x(t)$ decreases. This picture gives an idea of stability.



A fixed point is said to be stable is, for a small region $U(x^*)$ of a fixed point x^* all $x(t) \longrightarrow x^*$ if $x(0) \in U(x^*)$. There are more sophisticated definitions of stability which

we won't get into here. We can see from the picture that a fixed point is stable if

$$\left.\frac{df(x)}{dx}\right|_{x=x^\star} < 0$$

and unstable otherwise. If the derivative is exactly equal to zero, the situation is a bit more subtle.

1.3.4. Finding solutions analytically

In the 1d time continuous case one can always try to integrate the dynamical system, writing

$$\dot{x} = f(x) = \frac{dx}{dt}$$

and after separating variables

$$t - t_0 = \int_{x_0}^{x(t)} \frac{dx}{f(x)}$$

We can try to solve the right hand side of this equation, but have to be careful when we integrate across fixed points $f(x^*) = 0$. So for instance the Pearl-Verhulst equation

$$\dot{x} = \lambda x(1-x)$$

gives

$$\begin{aligned} \lambda(t-t_0) &= \int_{x_0}^{x(t)} \frac{dx}{x(1-x)} \\ \lambda(t-t_0) &= \log \frac{x(t)(1-x_0)}{(1-x(t))\,x_0} \end{aligned}$$

which yields

$$x(t) = x_0 \frac{e^{\lambda(t-t_0)}}{1 - x_0(1 - e^{\lambda(t-t_0)})}$$

which gives no more information than just drawing $f(x)$ as a function of x.

1.3.5. Finding (approximate) solutions numerically

Given dynamical system $\dot{\mathbf{x}} = f(\mathbf{x})$, ee can attempt to solve the dynamical system for an initial value $\mathbf{x}(t_0) = \mathbf{x}_0$ numerically. We split time into little discrete increments of length Δt, i.e. $t_n = t_0 + n\Delta t$, and $n = 0, 1, 2, ...$Then

$$\mathbf{x}(t_0 + \Delta t) \approx \mathbf{x}(t_0) + \Delta t f(\mathbf{x}(t_0))$$

Letting

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \Delta t\, f(\mathbf{x}_n) \quad \text{and} \quad \mathbf{x}_0 = \mathbf{x}(t_0)$$

we obtain an approximate solution. This is known as Euler's method. The accuracy is of der order Δt, that means if we for example pick $\Delta t = 0.01$ then after $N = 100$ steps the error will be or order unity which isn't so good. Much better are Runge Kutta methods, e.g. the second order Runge-Kutta method in which we advance the trajectory based on the dynamics at the midpoint of the interval. Let's call the state at the midpoint $\bar{\mathbf{x}}_n$ and the dynamics $f(\bar{\mathbf{x}_n})$. We then define

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \Delta t f(\bar{\mathbf{x}}_n)$$

But how do we find the midpoint? Well, we can't. Only approximately. We advance to the midpoint with ordinary Euler:

$$\bar{\mathbf{x}}_n = \mathbf{x}_n + \frac{\Delta t}{2} f(\mathbf{x}_n).$$

It's fairly straightforward to show that this method has an accuracy of the order Δt^2 so, again, if we let $\Delta t = 0.01$, we could integrate 10000 steps before the algorithm breaks down. Based on the same idea is the algorithm typically used in applications. It's accurate to 4^{th} order ($(\Delta t)^4$ error) and is called the RK4. The idea is just like the second order RK, building on functional values of many mid points:

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$K_1 = f(\mathbf{x}_n)\Delta t$$

$$K_2 = f(\mathbf{x}_n + \frac{1}{2}K_1)\Delta t$$

$$K_3 = f(\mathbf{x}_n + \frac{1}{2}K_2)\Delta t$$

$$K_4 = f(\mathbf{x}_n + K_3)\Delta t$$

Some of matlab's "ode" solvers use this method.

1.4. Bifurcations in 1d autonomous flows

Let's assume we have

$$\dot{x} = f(x; \mu)$$

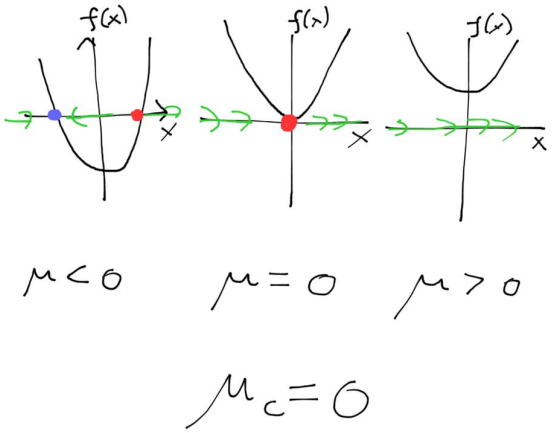
that only depends on one parameter μ. Since 1D flows are determined by the set of fixed points $\mathcal{F}$, we can use them to see how the behavior changes as μ is varied (i.e. how the set of fixed points and the stability changes).

1.4.1. Saddle node bifurcations

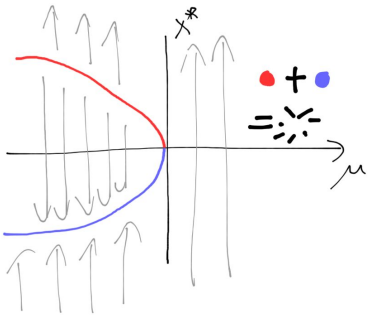
Let's study this in a simple model:

$$\dot{x} = f(x) = x^2 + \mu$$

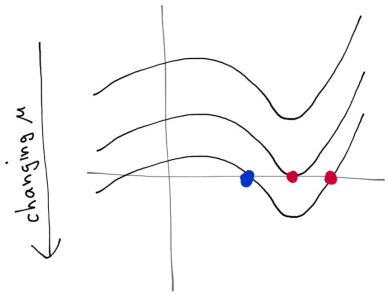
If we plot $f(x)$ for different values of μ we see that three different scenarios emerge:



This type of bifurcation is known as a **saddle-node bifurcation.** When $\mu < 0$ two fixed points exist, one is stable on is not. As μ is increased these fixed points approach each other and when $\mu = \mu_c = 0$ the system has one fixed point that attracts everything from the left but repels everything on the right. It's unstable. When μ is increased further, the fixed point disappears. in a sense, the stable and unstable fixed points annihilated. This is the bifurcation diagram, that tells use the dynamics as a function of μ.



A saddle-node bifurcation is fairly general. It always occurs when a parameter change induces an extremum of $f(x)$ to cross the abscissa like here:

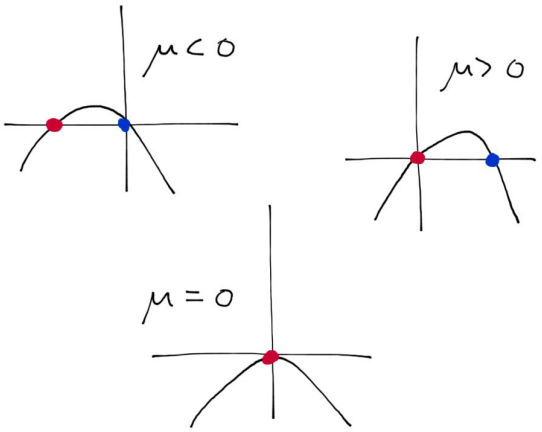


1.4.2. Transcritical bifurcation

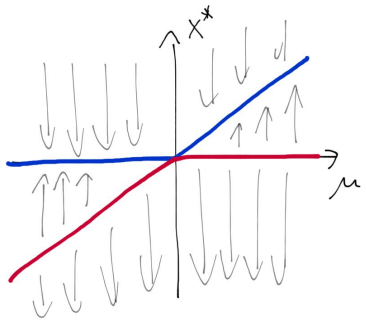
In addition to the saddle node bifurcation there's one called transcritical. This is when a stable and unstable fixed point collide and interchange their stability proerties. Here's and example:

$$\dot{x} = f(x;\mu) = \mu x - x^2 = x(\mu - x).$$

This guy always has a fixed point at $x = 0$ and one at $x = \mu$. However, for $\mu < 0$ this second fixed point is on the left of the origin and for $\mu > 0$ it's on the right. If we draw it if looks like this:



The red dot denotes an unstable fixed point, the blue one a stable fixedpoint. Ir we look at the bifurcation digram, we see that the fixed points collide and exchange their stability properties



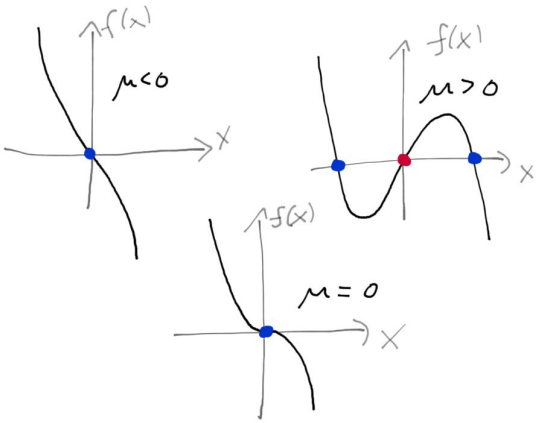
The two fixed points do not annihilate, but instead, exchange their stability. This is called a transcritical bifurcation.

1.4.3. Pitchfork bifurcation

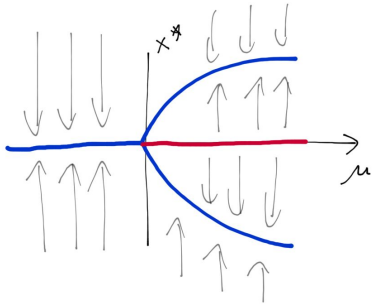
This is one that we discussed earlier. Let's look at

$$\dot{x} = f(x;\mu) = \mu x - x^3 = x(\mu - x^2).$$

Here we also have always a fixed point $x^\star = 0$. However, only for $\mu > 0$ we have two additional fixed pointx $x^\star = \pm\sqrt{\mu}$. If $\mu < 0$ they do not exist. Here's the picture that tells us what is going on:



What happens here is this. Let's start with $\mu < 0$. The system has a single stable fixed point at the origin. As we increase μ this stable fixed point gives birth to two new stable fixed points at $\mu = \mu_c = 0$ and loses its own stability. The bifurcation diagram looks like a pitchfork which is why this type of bifurcation is called what it's called:

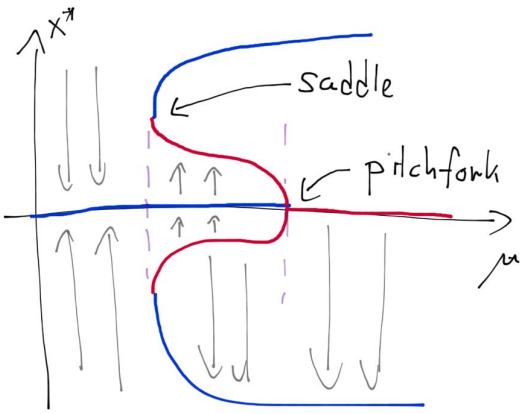


It can also happen that an unstable fixed point gives birth to two new unstable fixed points and in this process becomes a stable fixed point.

Saddle node, transcritical and pitchfork bifurcations are the most important bifurcations in 1d systems. There are more types of bifurcation but they are not as common.

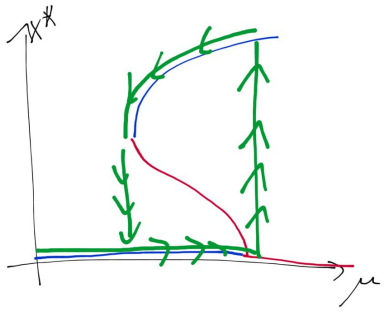
1.4.4. Hysteresis

So far, when we change parameters of a system that is in some equilibrium state $x^\star$ the value of this equilibirum changes slowly, even as we pass a critical point. However, the existance and interplay between bifurcations can yield some interesting behavior when many bifurcations exist. Without looking at any equations, let's imagine we have the following bifurcation diagram:



This is a system that has a stable fixed point at the origin for small values of μ. This point attracts all trajectories. As μ is increased to saddle node bifurcations occur and a regime emerges that has three stable fixed points (blue) and 2 unstable ones (red). Each of the three stable fixed points have a different attractor basin. Now imagine the system is in the $x = 0$ state and we perturb it a little. If the perturbation is small then the system will relax back to $x = 0$. However has we increase μ the system will undergo a pitchfork bifurcation and in response to any small pertubation (noise e.g.) the system

will make a giant leap to one of the other stable fixed points, because the $x = 0$ fixed point is no longer stable. This means that in response to a small parameter change in μ across the second critical value can yield a giant response. More interestingly, if I now decrease μ again below the second critical value, the system will not go back to the $x = 0$ state. I have to decrease μ until the saddle nodes disappear. At that stage the system will, again, make a giant leap back to $x = 0$ in response to a small change in μ. This effect is called hysteresis:



1.5. Two-dimensional systems

We discussed earlier that in 2D flows, we can have richer dynamics but asymptotically we can only have fixed points or periodic orbit, or limit cycles because of the Poincare Bendixon-Theorem. So let's briefly discuss systems that look like this

$$\begin{aligned} \dot{x} &= f(x, y) \\ \dot{y} &= g(x, y) \end{aligned}$$

The state trajectories evolve in the x-y-plane.

1.5.1. Nullclines

The first thing we do when we analyse 2D flows is trying to find all the fixed points, so points $(x^\star, y^\star)$ that fullfill

$$f(x^\star, y^\star) = g(x^\star, y^\star) = 0$$

Sometimes this is easy. Sometimes this is hard. One way to approach this graphically is by considering the nullclines, the $x-$nullcline and $y-$nullcline. These are curves in the plane that are implicitely given by

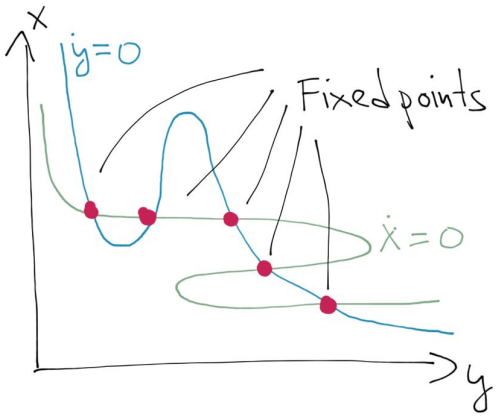
$$f(x, y) = 0$$

and

$$g(x, y) = 0,$$

these are curves in the plane which the solutions to the dynamical system have to cross parallel to the y and x axis, respectively and all the fixed points are given by their

intersection. If one is lucky one can express these nullclines as functions $y = \phi(x)$ or $x = \psi(y)$, but this is not always the case.



1.5.2. Stability of fixed points

In 1D systems we had a very simple condition for the stability of a fixed point given by the sign of the derivative

$$\left.\frac{df(x)}{dx}\right|_{x=x^\star}.$$

For stable and unstable fixed points this derivative has to be negative and postivie respectively. In 2D there's a similar condition, in fact we can consider a system in any dimension. So let's assume that we have a dynamical system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

with a fixed point $\mathbf{x}^\star$, so $\mathbf{f}(\mathbf{x}^\star) = 0$. Now let's assume we start the system off in the vicinity of the fixed point and evolve it

$$\mathbf{x}(t) = \mathbf{x}^\star + \epsilon\mathbf{y}(t)$$

where $\varepsilon \ll 1$ and $\mathbf{y}(0) = \mathbf{y}_0$. Now, we are interested in what happens to $\mathbf{y}(t)$

$$\begin{aligned}\frac{d\mathbf{x}}{dt} &= \varepsilon\frac{d\mathbf{y}}{dt} = \mathbf{f}(\mathbf{x}^\star + \varepsilon\mathbf{y}(t)) \\ &\approx \mathbf{f}(\mathbf{x}^\star) + \varepsilon\mathbf{J}(\mathbf{x}^\star)\mathbf{y}(t)\end{aligned}$$

and thus

$$\dot{\mathbf{y}} = \mathbf{J}(\mathbf{x}^\star)\mathbf{y}$$

in the vicinity of the fixed points. the matrix $\mathbf{J}(\mathbf{x}^\star)$ is the Jacobian of $\mathbf{f}(\mathbf{x})$ at the fixed point. The elements of the matrix are given by

$$\mathbf{J}_{ij}(\mathbf{x}^\star) = \frac{\partial f_i(\mathbf{x})}{\partial x_j}.$$

So for example, let's say we have a dynamical system

$$\begin{aligned}\dot{x}_1 &= x_2 = f_1(x_1, x_2)\\ \dot{x}_2 &= x_1(1-x_1) - x_2 = f_2(x_1, x_2)\end{aligned}$$

This system has a fixed point at $(x_1, x_2) = (1, 0)$ and at the origin. The Jacobian is given by

$$J(\mathbf{x}) = \begin{pmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1-2x_1 & -1 \end{pmatrix}$$

which is for the first fixed point

$$J(\mathbf{x} = (1,0)) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$$

and for the second fixed point

$$J(\mathbf{x} = (0,0)) = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$$

Let's look at the above equation again:

$$\dot{\mathbf{y}} = \mathbf{J}\mathbf{y}$$

This is a linear system of equations. And for those we know from ODE theory that this has a solution for which $\mathbf{y}(t) \to 0$ if all the eigenvalues have negative real parts. So, if we have a fixed point $\mathbf{x}^\star$ and the Jacobian $\mathbf{J}(\mathbf{x}^\star)$ and a set of eigenvalues $\lambda_1, ..., \lambda_2$ if all of those have negative real parts then the fixed point is asymptotically stable. If on the other hand, the real part of at least one eigenvalue is positive, the fixed point is not stable, because trajectories leave the fixed point for almost all perturbations.

1.5.2.1. Solving the linear system

In order to understand a bit better the condition of the eigenvalue's consider the following approach. Let's make an ansatz for a solution to

$$\dot{\mathbf{y}} = \mathbf{J}\mathbf{y}$$

given by

$$\mathbf{y}(t) = \mathbf{y}_0 e^{\lambda t}$$

Plugged into the above equation gives

$$\lambda \mathbf{y} = J\mathbf{y}$$

so, this is a solution if $\mathbf{y}$ is an eigenvector of $\mathbf{J}$ with eigenvalue λ. This particular solution will only tend to zero, if $Re(\lambda) < 0$. Now, generically we have in an n-dimensional system n linearly independent eigenvectors with corresponding different eigenvalues. Generically.

Sometimes different eigenvectors exist for the same eigenvalue. Let's not consider those for th moment. If these eigenvectors $\mathbf{y}_k$ with $k = 1, ..., n$ are linearly independent they form a basis in phase space and any solution to the above ODE is given by a linear combination

$$\mathbf{y}(t) = \sum_{k=1}^{n} c_k \mathbf{y}_k(t) = \sum_{k=1}^{n} c_k \mathbf{y}_k(0) e^{\lambda_k t}$$

which means that this solution can only tend to zero if ALL the eigenvalues' real parts are negative.

1.5.2.2. Eigenvalues of a 2D matrix

If we have a 2D system the Jacobian is going to be a 2D matrix, say

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

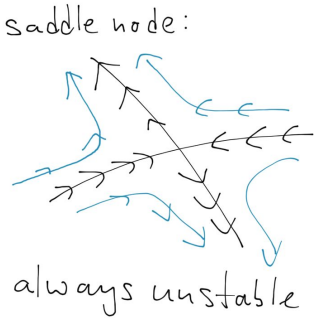
Let's call $s = a + d$ the trace of this matrix and $\Delta = ad - bc$ its determinant. The eigenvalues of a 2D matrix are given by

$$\lambda_{1/2} = s/2 \pm \sqrt{s^2/4 - \Delta}$$

so, in order to have a stable fixed point we must have $s < 0$ and $\Delta > 0$. Let's look at the above example. For the first we have $s = -1$ and $\Delta = 1$ so that fixed point is stable. For the other one we get $s < 0$ but $\Delta < 0$ so that one is unstable. Depending on the values for the trace, one can categorize fixed points into the following subtypes

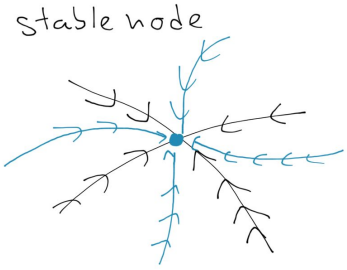
1.5.2.3. Saddle nodes $\Delta < 0$

These are unstable fixed points for which one eigenvalue is positive, the other negative, and the local dynamics looks like this:



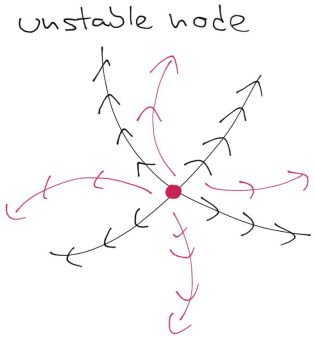
1.5.2.4. Stable node $0 < \Delta < s^2/4$ and $s < 0$

In this case both eigenvalues are negative and in the vicinity of the fixed point the dynamics looks like this



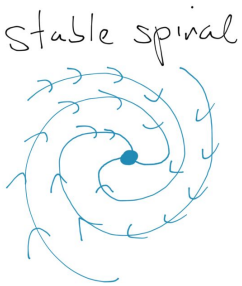
1.5.2.5. Unstable node $0 < \Delta < s^2/4$ and $s < 0$

In this case both eigenvalues are positive and in the vicinity of the fixed point the dynamics looks like this



1.5.2.6. Stable spiral: $s^2/4 < \Delta$ and $s < 0$

if this is the case, the expression in the $\sqrt{}$ becomes negative and the eigenvalues are complex and complex conjugates to one another, with a negative real part. The imaginary part introduces oscillatory behavior, the negative real part damps it down. We have a stable spiral:

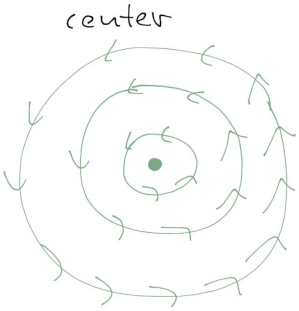


1.5.2.7. Unstable spiral $s^2/4 < \Delta$ and $s > 0$

This is just the same, but the oscillatory behavior is ammplified, to the spiral looks like this:

1.5.2.8. An interesting case is when $s = 0$ and $\Delta > 0$

In this case the eigenvalues are purely imaginary and in the vicinity of the fixed point we may expect periodic orbits. These fixed points are called centers:



So, in summary, we first have to compute the nullclines, see where they intersect, then we've got the fixed points. Next we compute their stability and then we already have a pretty good grasp of what's going on.

1.5.3. Example: Damped Duffing Oscillator

Let's look at this system which will become important later.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -y + x - x^3 \end{aligned}$$

the nullclines are given by the two curves $\phi_x(x) = 0$ (the x-axis) and $\phi_y(x) = x - x^3$ which intersect at 3 different points, (0,0),(-1,0),(1,0), so there are three fixed points. For

the stability, we have to investigate the Jacobian at these points. The Jacobian is given by

$$\mathbf{J}(x,y) = \begin{pmatrix} 0 & 1 \\ 1-3x^2 & -1 \end{pmatrix}$$

So the fixed point $(0,0)$ has trace $s=-1$ and $\Delta=-1$ which means the fixed point is unstable. The opposite is true for the other fixed points because for those $\Delta=1$. In fact thise means that the origin is a saddle, and the other one are stable spirals.

1.5.4. Periodic Solutions

One of the things that can exist in 2D systems are periodic solutions. Let's look at this system

$$\begin{aligned} \dot{x} &= x(b-py) \\ \dot{y} &= y(rx-d) \end{aligned}$$

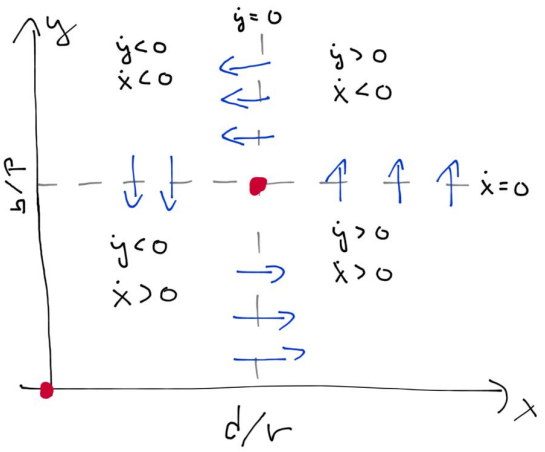
with constants $b,p,r,d>0$ This system is known as the **Lotka-Volterra system**. It can be used to model the dynamics of a two-species ecosystem where $x(t)$ is the number of prey animals and $y(t)$ a predator species. Both equations say that both species reproduce at rates that depend on the concentration of the other species. The rate of reproduction of the prey is given by $b-py$. This means that there's a constant reproduction rate that is decreased with the number of predators. The predator reproduction rate is $rx-d$ which means that predators reproduce proportionally to the number of prey and die at a constant rate d. First thing we do is try to find fixed points. These are given by

$$(0,0) \quad \text{and} \quad (d/r, b/p).$$

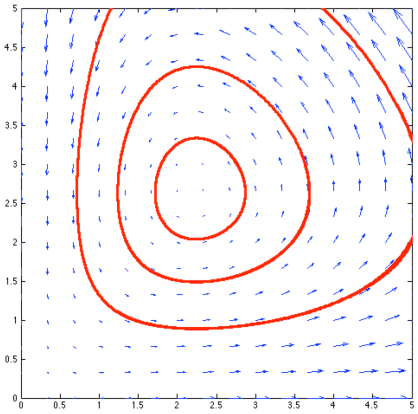
Here's something to interpret already. The trivial fixed point is clear. No species is a stationary solution. The other says, in this this equilibrium the number of prey increases with the death rate of the predator and decreases with the reproduction rate r of the predator. The predator number increases with the birth rate of the prey and decreases with the predation rate p.

1.5.5. Plotting Nullclines

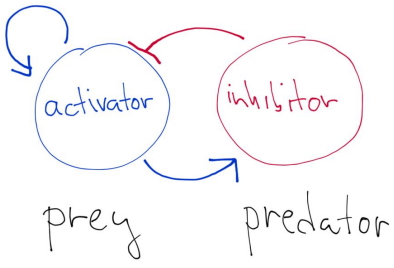
The next thing we do, in order to get an idea about the dynamics, is plot the nullclines of x and y. These are curves in where $\dot{x}=0$ and $\dot{y}=0$, respectively, called the x-nullcline and y-nullcline. The nullclines meet (obviously) in the fixed point. Trajectories of the system have to traverse the x-nullcline parallel to the y-axis the opposite is the case for the y-nullcline. The nullclines split (in this system) phase space in 4 regions defined by the sign of $\dot{x}$ and $\dot{y}$. By drawing arrows on the nullcline crossings we already see that this system has some potential for periodic orbits:



In fact, if we compute some trajectories numerically we see that different periodic orbits exists depending on the initial condition consistent with the overall picture that is suggested by the nullclines:



The potential of oscillations in this system is very generic in systems with two interacting quantities that follow an activator inhibitor scenario. In this particular case, we can think of the predator inhibiting the prey, and the prey activating the predator and themselves. This is the mechanism that produces oscillations:



1.5.6. Example: Duffing Oscillator

Let's look at the following dynamical system

$$\ddot{x} = \beta x - \alpha x^3 - \delta \dot{x}$$

where $\alpha, \beta, \delta > 0$ are parameters. This is the dynamic equation that describes the motion of an oscillator subject to a force

$$f(x) = \beta x - \alpha x^3$$

This force means that the oscillator is repelled from the origin by the linear term and attraced by the cubic term. The term $-\delta \dot{x}$ describes the friction in the system. We can write this equation as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \beta x - \alpha x^3 - \delta y \end{aligned}$$

1.5.6.1. Fixed points and their stability

Again, we first find the fixed point all of which will have $y = 0$ (x-nullcline) and $x = 0$ and $x = \pm\sqrt{\beta/\alpha}$. We find the stability using the Jacobian

$$J(x, y) = \begin{pmatrix} 0 & 1 \\ \beta - 3\alpha x^2 & -\delta \end{pmatrix}$$

At the origin this is

$$J(0, 0) = \begin{pmatrix} 0 & 1 \\ \beta & -\delta \end{pmatrix}$$

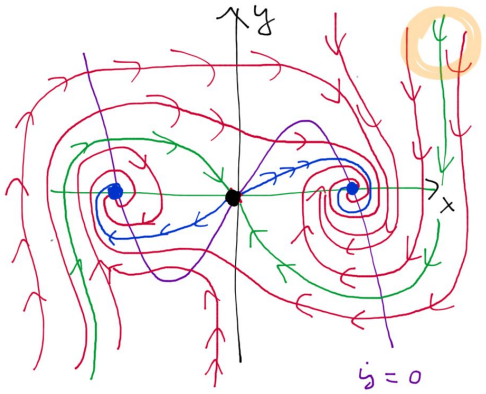
with trace $s = -\delta$ and determinant $\Delta = -\beta$. So this fixed point is a saddle (unstable) for all parameters. For the other fixed points we get

$$J(\pm\sqrt{\beta/\alpha}, 0) = \begin{pmatrix} 0 & 1 \\ -2\beta & -\delta \end{pmatrix}$$

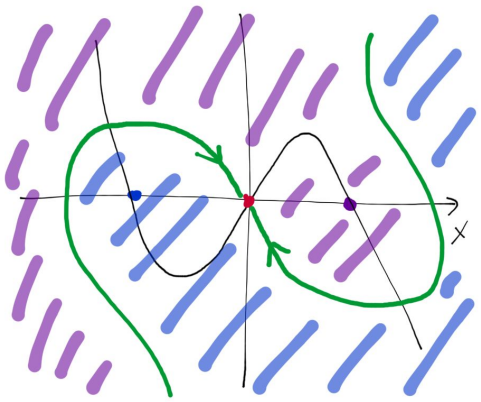
so the traces is $s = -\delta$ and the determinant is$\Delta = 2\beta > 0$ which means that these fixed points are stable. Recall that when

$$\begin{aligned} \Delta &> s^2/4 \\ \beta &> \delta^2/8 \end{aligned}$$

we have stable spirals, because the eigenvalues have a complex component. The overall behavior in phase space looks like this:

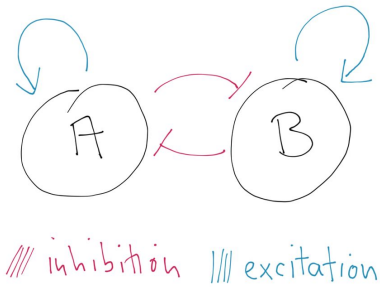


Note the stable spirals on either side and the saddle node in the middle. The green line is the attractive part of that node and it separates the basins of attraction of the other two fixed points. Here's something to observe in this system which will be essential when we investigate this system as a potential for chaotic dynamics. Note the orange region in the top right. We can have initial conditions that are very close, in this case near to the green line, but they have very different fate, their trajectories separate, which is why the green line is called separatrix. So there is divergence in fate. On the other hand, for different initial conditions that are far apart we can hape the opposite effect, convergence to one of the fixed points. This combination of separation and convergence at play at the same time is very important for chaotic motion in dissapative, driven systems, which we will discuss using this example in more detail. Here's a picture of the attractor basins:



1.5.7. Example: Gene regulation

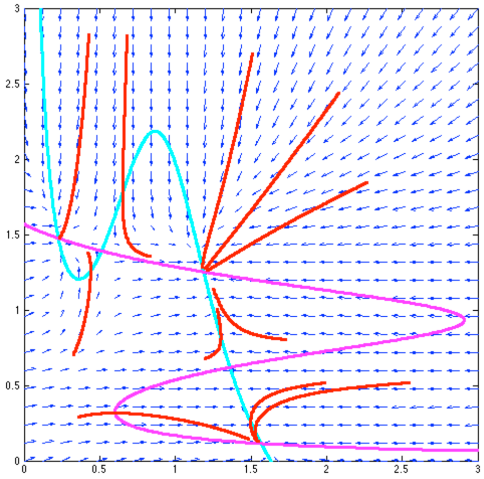
In developmental biology it is of interest to understand how genes regulate each other. In a cell, a gene A and express a protein which can enhance or inhibit the expression of other genes, including gene A itself. Because of this mechanism, it is possible that cells with the same genome have different phenotypes, for instance liver cells, stem cells, pancreatic cells, skin cells etc. One mechanism that is very important is the interaction of two genes, say A and B that enhance their own expression but inhibit each other:



Let's denote by x and y the expression levels of gene A and B, respectively. We can model this as a dynamical system

$$\begin{aligned} x &= -\gamma x + \alpha \frac{x^2}{1+x^2} + \beta \frac{1}{1+y^2} \\ y &= -\gamma y + \gamma \frac{y^2}{1+y^2} + \delta \frac{1}{1+x^2} \end{aligned}$$

The first term in each equation model a constant decay, taking account of the effect that proteins break down. The second term (first equation) means that the higher the expression level, $x \gg 1$, the higher the input, i.e. self excitation. The third term accounts for the inhibition, if $y \gg 1$ that close to zero, much smaller than when $y \ll 1$. Depending on the parameters this system can have between 1 and 3 stable fixed points. Here are two phase plots corresponding to a parameter set that has three stable fixed points. In one, both x and y are high, and in the other two one gene expression is high and the other is low:



2. Random Walks and Diffusion

In order to understand the dynamics of complex systems we have to understand the basics of stochastic processes. This art is concerned with the description of random time-evolution. It is essentially probability theory as a function of time. So instead of random numbers, stoch. processes are random variables $X(t)$ that change over time. And again, sometimes time is continuous and sometimes discrete, so a process is a sequence X_n of random numbers.

2.1. Simple Random Walks

The best way to introduce random motion is by investigating a random walk process. This is a process where at each discrete time step n a randomly chosen increment gets added to the position of the walker

$$\mathbf{Y}_{n+1} = \mathbf{Y}_n + \mathbf{X}_n$$

where the starting point $Y_0 = x_0$ is usually taken to be the origin, i.e. $x_0 = 0$. And the increments X_n are each drawn from the same probability distribution $p(x)$ and it is assume that the increments are independent. The above equation can be "integrated" and gives

$$Y_n = \sum_{k=0}^{n} X_k.$$

Let's assume that the increments have zero mean

$$\langle X \rangle = \int dx\, x\, p(x).$$

and some variance

$$\left\langle X^2 \right\rangle = \int dx\, x^2\, p(x) = \sigma^2.$$

Let's compute the mean and variance of Y_n:

$$\langle Y_n \rangle = \left\langle \sum_{k=0}^{n} X_k \right\rangle = \sum_{k=0}^{n} \langle X_k \rangle = 0$$

and

$$\begin{aligned}
\langle Y_n^2 \rangle &= \left\langle \sum_{k,l} X_k X_l \right\rangle = \sum_{k,l} \langle X_k X_l \rangle \\
&= \sum_{k=l} \langle X_k X_l \rangle + \sum_{k \neq l} \langle X_k X_l \rangle \\
&= \sum_{k=1}^{n} \langle X_k^2 \rangle + \sum_{k \neq l} \langle X_k \rangle \langle X_l \rangle \\
\langle Y_n^2 \rangle &= n\sigma^2
\end{aligned}$$

So the variance increases with stepnumber n. This also means that the typical distance from the starting point is

$$\sqrt{\langle Y_n^2 \rangle} = \sigma\sqrt{n}.$$

That means for 4 times the number of steps, the walker only doubles it expected distance. This is true, irrespective of the shape of the pdf for the single steps $p(x)$. Let's discuss a few examples of a random walk on a line, i.e. a one dimensional random walk

1. Let's assume that at each time step the walker makes a step that is drawn from a gaussian distribution with zero mean and variance σ^2

$$p_1(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}x^2/\sigma^2}$$

2. We could also have a random walker that makes steps of fixed length, but random direction

$$p_2(x) = \frac{1}{2}\left[\delta(x-\sigma) + \delta(x+\sigma)\right]$$

where $\delta(x)$ is the Dirac delta function which is defined as

$$\delta(x) = \begin{cases} 0 & \text{for} \quad x \neq 0 \end{cases}$$

and

$$\int dx \delta(x) = 1,$$

essentially this is a function that is everywhere zero, except for at the origin, where it is infinite. Another property of it is that

$$f(y) = \int dx\, \delta(y-x) f(x).$$

so that for this pdf we have

$$\langle X^2 \rangle = \frac{1}{2} \int dx\, x^2 \left[\delta(x-\sigma) + \delta(x+\sigma)\right] = \sigma^2$$

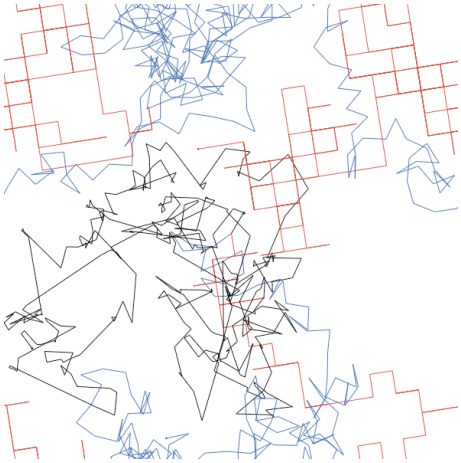
3. Finally let's consider a pdf that has exponentially distributed steps

$$p_3(x) = \frac{1}{2\sigma} \exp\left[-|x|/2\sigma\right]$$

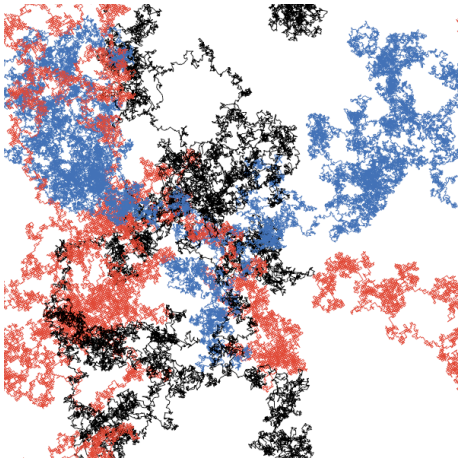
In this case we have

$$\langle X^2 \rangle = \frac{1}{\sigma} \int_0^\infty dx\, x^2\, e^{-x/2\sigma} = \sigma^2$$

So all of these three random walks look different. Here are three pictures of two dimensional variants generated with netlogo:



We clearly see, e.g. that the read trajectory is different from the other two but those do also differ in shape. In the red trajectory, the walker goes back, forward, left and right with equal probability. In the black, the walker picks a random direction and then makes a step of a length that is exponentially distributed, whereas in the blue the walker picks a distance that has a Gaussian distribution. These distributes are gauged so the single step variance is the same. Now let's look at this at a larger scale for a longer time:



At this scale, the difference are no longer visible, or barely visible. If we go to larger scales, the trajectories become statistically indistiguishable. This brings us to and important result called the....

2.1.1. Central Limit Theorem

The sum

$$Y_n = \sum_{k=1}^{n} X_k$$

is a sum of random numbers. The question is, what is the distribution of Y_n and how does it behave as $n \to \infty$. The way one can be a bit more precise is to define

$$Z_n = \frac{Y_n}{A(n)}$$

and ask what is the limiting distrubution of Z_n and for what scaling function $A(n)$. The answer to this is, if we chose $A(n) = \sigma n$ then

$$p(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2},$$

the normal distribution. In more intuitive terms this means: The sum of random numbers approaches a normal distribution as $n \to \infty$, this is why the above processes look the same in the limit of $n \to \infty$.

2.1.2. Time continuous limit

Now, the above random walk approach was derived on discrete time. Let's now see if we can derive a time continuous version of it. Let's assume that we have a time intervale 0 to t and that we split it into n intervals of length Δt so

$$t_n = n\Delta t \quad n = 1, ..., N$$

Let's identify the position of the walker at step N with $Y(t)$. so

$$Y(t) = Y_N = \sum_{n=1}^{N} X_n.$$

We know that the variance of $Y(t)$ is

$$\langle Y(t)^2 \rangle = N\sigma^2.$$

We would now like to perform the limit $N \to \infty$ and $\Delta t \to 0$ keeping the typical distance from the origin only as a function of time (which we consider fixed for now)

$$d = \sqrt{\langle Y(t)^2 \rangle} = \sqrt{N\sigma} = \sqrt{t\frac{\sigma}{\Delta t}}.$$

This only works if we let the parameter of the single step length σ go to zero as

$$\sigma = D\Delta t$$

because in this case

$$d(t) = \sqrt{Dt}$$

So the trajectories

$$Y(t) \sim \sqrt{Dt}$$

separate from the distance as the square root of t. Essential this is what the process does. If a large number of very small impacts influence the position of say a particle, then these will move a particle a typical distance from the origin proportional to $\sqrt{t}$ and not t.

2.1.3. Adding noise to a deterministic system

Now let's assume just that. Let's assume we have a deterministic system that evolves according to

$$\dot{x} = f(x)$$

For simplicity we consider a 1-d system. That means that after a small time interval Δt the position is given by

$$x(t + \Delta t) = x(t) + \Delta t \times f(x(t))$$

Now let's assume that during the time Δt, and very large number of small impacts also moved the particle, each impact moving it equally likely in either direction. This will add a contribution

$$\Delta W(\Delta t) \sim \sqrt{\Delta t}$$

so addind both components gives

$$X(t + \Delta t) = X(t) + \Delta t \times f(X(t)) + \sigma \Delta W$$

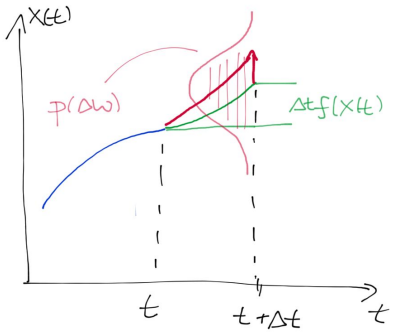
which we can also write as

$$dX \quad = \quad dt\, f(X) + \sigma dW$$

This is called a Langevin equation. What we mean by

$$X(t + \Delta t) = X(t) + \Delta t \times f(X(t)) + \sigma \Delta W(\Delta t)$$

is that in a small time interval, the deterministic force pushes the state a bit forward by $\Delta t \times f(X(t))$ and the random influence adds a Gaussian random increment ΔW which has a variance of Δt so is or order $\sqrt{\Delta t}$. The prefactor σ is the magnitude of this impact. Here's the picture:



2.1.3.1. Example: System in a double well

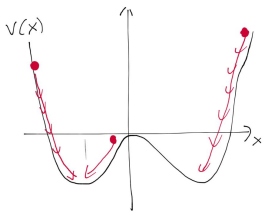
Let's look at the following 1d system:

$$\dot{x} = -\frac{dV(x)}{dx}$$

This is called a gradient system, and $V(x)$ is a potential. This means the dynamical system has fixed points where the potential has extrema, i.e. maxima and minima. Because of the negative sign, x will increase if the slope of $V(x)$ is negative and decrease if the slope is positive, this means the minima of $V(x)$ are stable fixed points and the maxima are unstable fixed points. So let's look at

$$V(x) = \frac{1}{4}ax^4 - \frac{1}{2}x^2 = \frac{1}{4}x^2(ax^2 - 2)$$

This potential has a maximum at 0 and two minima at $\pm 1/\sqrt{2/a}$.



The dynamical system reads

$$\dot{x} = x - ax^3$$

Now let's assume that the system also has a noise force which means

$$dX = (X - AX^3)dt + \sigma dW$$

so at each time step Δt in addition to the determinstic evolution we add a gaussian random variable ΔW with variance Δt and multiply it by a constant σ. So we draw a random number from the probability distribution

$$p(\Delta w) = \frac{1}{\sqrt{2\pi\Delta t}} e^{-\frac{1}{2}\Delta w^2/\Delta t^2}$$

Let's look at what happens in the analogous system in two dimensions. Let's look at a system which is governed by a potential

$$V(\mathbf{x}) = \frac{1}{4}a\mathbf{x}^4 - \frac{1}{2}\mathbf{x}^2$$

and

$$\dot{\mathbf{x}} = -\nabla V(\mathbf{x})$$

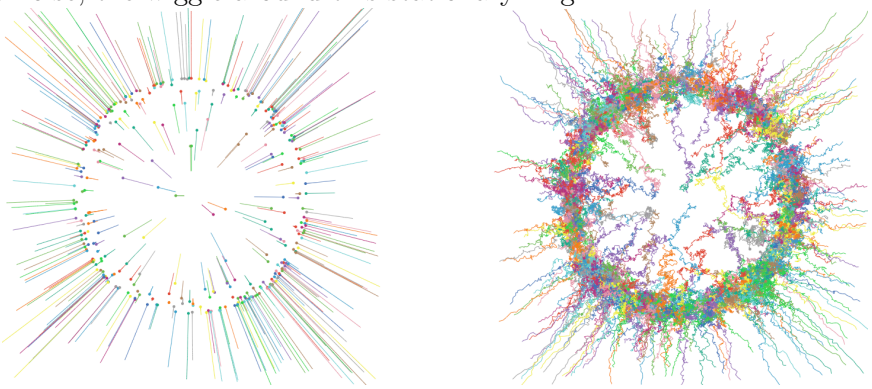
which in components reads

$$\begin{aligned} \dot{x} &= -ax^3 - axy^2 + x \\ \dot{y} &= -ay^3 - ayx^2 + y \end{aligned}$$

Now let's add noise, so

$$\begin{aligned} dx &= \left[-ax^3 - axy^2 + x\right] dt + \sigma dW_x \\ dy &= \left[-ay^3 - ayx^2 + y\right] dt + \sigma dW_y \end{aligned}$$

so at each time step we add two independent gaussian increments. Without the noise the trajectories will move towards the origin and get stuck at a distance $\sqrt{a/2}$. If we add a little noise, the wiggle around this stationary ring:



2.1.4. The Wiener Process

The simplest process is one in which the position or state of a system is just determined by the random gaussian increments ΔW so

$$W(t) = \int_0^t dW$$

by which we mean that we are adding an infinite amount of Gaussian increments of variance dt, so the limiting process of

$$W(t + \Delta t) = W(t) + \Delta W$$

where the ΔW have a Gaussian distribution with variance Δt. This process is called the Wiener process. Of course the process has to start somewhere, say at w_0. So each

trajectory $W(t)$ has the same initital condition $W(0) = w_0$, but obviously each trajectory has a different fate. Now, because the increments have a Gaussian distribution with variance Δt then so does $W(t)$ with variance t. The process $W(t)$ can thus be described by the probability $p(w, t|w_0, t_0)$ which is the proability that $W(t) = w$ conditioned on $W(t_0) = w_0$. It turns out that in this case

$$p(w|t, w_0, t_0) = \frac{1}{\sqrt{2\pi(t - t_0)}} e^{-\frac{1}{2}(w-w_0)^2/(t-t_0)}$$

which is a Gaussian that gets broader and broader over time. We can think of $W(t)$ as a randomly moving particle that started at position w_0 at $t = 0$. Let's assume $t_0 = 0$ and $w_0 = 0$ for now and write for the $p(w, t) = p(w, t|w_0, t_0)$, so

$$p(w, t) = \frac{1}{\sqrt{2\pi t}} e^{-w^2/2t}.$$

Now a key thing is that because each realization of the process is different and random, it makes little sense to define the state of the system as the value of $W(t)$ for one realization. In fact, what makes more sense, is to define the state of the system by the pdf $p(w, t)$ which changes over time, too, but deterministically. In fact we can write down a dynamical equation for it

$$\partial_t p(w, t) = \frac{1}{2} \partial_w^2 p(w, t).$$

We can learn a few more things from the pdf dynamics. For example, what's the mean

$$\langle W(t) \rangle = \int dw \, w \, p(w, t) = 0$$

The typical distance from the starting point is given by $d(t) = \sqrt{W^2(t)}$

$$\begin{aligned} \langle W(t)^2 \rangle &= \frac{1}{\sqrt{2\pi t}} \int dw \, w^2 \, e^{-w^2/2t} \\ &= t \end{aligned}$$

which means that

$$d(t) = \sqrt{t}$$

as expected.

2.1.5. The state of a stochastic system

Remember that we discussed that in a dynamical system we can predict the future knowning the history of a system. Of course in a stochastic system this is impossible due to the internal unpredictability of the system. Every's process $X(t)$ is different. This is why it would be unhelpful to describe the state of the system by $X(t)$. Instead, in a stochastic process we describe the state of the system by the probability $p(x, t)$ of the

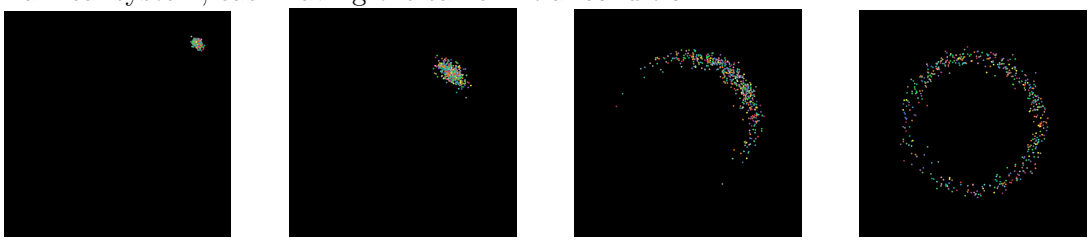
system having a value $X(t)$. This, unlike the variable $X(t)$ is predictable and evolves according to a determinstic equation, e.g. the diffusion equation above

$$\partial_t p = \frac{1}{2} D \partial_x^2 p.$$

This also changes our notion of equilibrium. Let's say we have a process like the one above, in which the dynamics are governed by

$$\begin{aligned} dx &= \left[-ax^3 - axy^2 + x\right] dt + \sigma dW_x \\ dy &= \left[-ay^3 - ayx^2 + y\right] dt + \sigma dW_y \end{aligned}$$

Here's a picture of the trajectories of a bunch of particles that evolve according to this dynamical system, each having the same initial condition:



As time evolves the distribution of particles does no longer change and moves to a distribution which is given by

$$p(x, y) = \frac{1}{\mathcal{N}} \exp[-\frac{1}{4\sigma^2} \left(\mathbf{x}^4 - 2\mathbf{x}^2\right)].$$

and the dynamical equation that governs this is known as the Fokker-Planck equation which reads

$$\partial_t p(\mathbf{x}, t) = -\nabla \cdot F(\mathbf{x}) p(\mathbf{x}, t) + \frac{1}{2} \sigma^2 \Delta^2 p(\mathbf{x}, t)$$

with an initital distribution of particles given by $p(\mathbf{x}_0, t_0)$ and $F(\mathbf{x}) = -\nabla V(\mathbf{x})$. It is outside the bounds of this course to discuss how this equation is derived. Here it's sufficient that it is possible to derive a dynamical system for the time evolution of this probability function.

2.1.6. Particle kinetics and Jump processes

In the above examples, the picture was that we had a determinstic system subject to external noise which took shape in a gaussian pertubation that constantly exerts a random change in the trajectory of the system. There are however, other systems in which randomness plays a role and which cannot be directly modeled like this. Consider a chemical reaction of two molecules A and B or rather two chemical reactions

$$\begin{aligned} A + B &\xrightarrow{\alpha} 2A \\ A + B &\xrightarrow{\beta} 2B \end{aligned}$$

The first reaction means that if two molecules meet, one is converted to the other at a rate α and the second means the conversions is to B at a rate β. Let's say we have N molcules, so $n_A + n_B = N$. So what we first notice is that this number does not change, only the relative components. The dynamic variable that describes the system is the number of A particles (because the number of B particles is given by $n_b = N - n_a$). Let's assume that these two reaction occur randomly, and let's assume that at time t the system is in a state $n_A(t) = n(t)$. The three things can happen in a small time interval Δt:

1. $n(t + \Delta t) = n(t)$ nothing happens with probability p_0
2. $n(t + \Delta t) = n(t) + 1$ which means reaction 1 occured with probability p_+
3. $n(t + \Delta t) = n(t) - 1$ reaction 2 occured with probability p_-

Because of this, these processes are called jump processes. Now we need to make assumptions on the probability that each one of the events occured in a small time-interval Δt, we must have

$$p_0 + p_+ + p_- = 1$$

Now in general all probabilities will depend on the number N_A N and Δt. Let's assume that

$$\begin{aligned} p_+ &= \Delta t \alpha n_A n_B / N = \Delta t \times w_+(n) \\ p_- &= \Delta t \beta n_A n_B / N = \Delta t \times w_-(n) \\ p_0 &= 1 - \Delta t \times [w_+(n) + w_-(n)] \end{aligned}$$

Because $n_B = N - n_A$ we can write this as

$$\begin{aligned} p_+ &= \Delta t \alpha n(N - n)/N = \Delta t \times w_+(n) \\ p_- &= \Delta t \beta n(N - n)/N = \Delta t \times w_-(n) \\ p_0 &= 1 - \Delta t \times [w_+(n) + w_-(n)] \end{aligned}$$

Now let's assume that we have an ensemples of such systems evolve over time and it's in a state $p(n, t)$ which measures the probability of seeing exactly n particles of type A in a system. How does this probability change over time

$$p(n, t + \Delta t) = p_0(n)p(n, t) + p_+(n - 1)p(n - 1, t) + p_-(n + 1)p(n + 1, t)$$

which gives

$$\partial_t p(n, t) = w_+(n - 1)p(n - 1, t) + w_+(n + 1)p(n + 1, t) - [w_+(n) + w_-(n)]\, p(n, t)$$

This is called a Master equations. It tells us how the probability changes over time. In the particular case above it reads

$$\partial_t p(n,t) = \alpha \frac{(n-1)(N-n+1)}{N} p(n-1,t) + \beta \frac{(n+1)(N-n)}{N} n\, p(n+1,t) - (\alpha+\beta)\frac{n(N-n)}{N} p(n,t)$$

These type of equations are generally difficult to solve. However one get get an idea of how the mean $\langle n(t) \rangle$ over time

$$\begin{aligned} \partial_t \langle n(t) \rangle &= \frac{d}{dt} \sum_{k=0} k\, p(k,t) \\ &\quad \sum_{k=0} k\, dp(k,t)/dt \\ &= \langle w_+(n(t)) \rangle - \langle w_-(n(t)) \rangle \end{aligned}$$

In the case above this means

$$\partial_t \langle n \rangle = (\alpha - \beta) \left[\langle n \rangle - \frac{\langle n^2 \rangle}{N} \right]$$

so if we let $x = n/N$ then

$$\partial_t \langle x \rangle = (\alpha - \beta) \left[\langle x \rangle - \langle x^2 \rangle \right]$$

This looks almost like

$$\partial_t x = (\alpha - \beta) \left(x - x^2 \right)$$

the equation we would have expected if no randomness is involved. However, randomness is involved and generally $\langle x^2 \rangle \neq \langle x \rangle^2$.

Instead of trying to solve the master equation one typically uses the algorithm to generate trajectories that follow the mechanics of the problem. One has to remember only that the system stays in its state for some time T and then moves to another state. So, in order to simulate a trajectory we have to know what the time distribution is for staying in a state, and the probabilities of going up or going down. Given the system is in a state $n(t)$ the probabililty that it stays in that state for a short time is given by

$$p_0(\Delta t, n) = 1 - \Delta t \times \left[w_+(n) + w_-(n) \right].$$

Let's make this equation a bit simpler by introducing

$$\gamma(n) = w_+(n) + w_-(n)$$

This is the rate at which the system changes a state.

The probability that it stays in that state for exactly a time T and then leaves the state is computed the following way. We say that $T = k\Delta t$ then

$$p(T,n) \approx \left(1 - \Delta t \times \left[\gamma(n)\right].\right)^k \gamma(n) \Delta t.$$

which becomes

$$p(T, n) = \gamma(n)e^{-T\gamma(n)}dt$$

So the probability distriubtion for the waiting times is an exponential the rate parameter $\gamma(n)$ of which depends on n. But after time T into which state is the system going to go? This is given by

$$Q_+ = \frac{w_+}{w_+ + w_-} \quad \text{and} \quad Q_+ = \frac{w_-}{w_+ + w_-}$$

This means that we can simulate this system in the following way.

1. First we set the initial conditions $n(0) = n_0$
2. Then we draw a random waiting time from the expoential distribution

$$p(T, n(0)) = \gamma(n(0))e^{-T\gamma(n_0)}$$

3. then we compute the next state according to Q_+ and Q_- and set $n(T)$ to the next state.

Let's look at the example above again. We have

$$w_+(n) = \alpha n(1-n) \quad w_-(n) = \beta n(1-n)$$

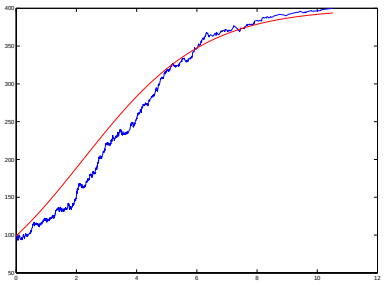
so

$$\gamma(n) = (\alpha + \beta)n(1-n)$$

and

$$Q_+ = \frac{\alpha}{\alpha + \beta} \quad Q_- = \frac{\beta}{\alpha + \beta}$$

And this is what a trajectory looks like:



2.1.7. Example: Generalized predator prey systm

Let's see if we can use this approach to model a predator prey system in which we also have two populations. Let's assume we have a density n_A and n_B of prey and predators respectively and let's assume that these are discrete numbers, so the count of the two

populations. Let's assume that the following reactions can occur at different per capita rates

$$\begin{aligned} A + B &\xrightarrow{\alpha} 2B \\ B &\xrightarrow{\beta} \emptyset \\ A &\xrightarrow{\gamma} 2A \\ 2A &\xrightarrow{\delta} A \end{aligned}$$

The first reaction models that the predator eats prey and reproduces. The second one models the death of a predator, the third equation means the spontaneous reproduction of prey, and the third could model competition for resources among the prey populations. If we were to set up a determinstic system for this, we would end up with odes that read

$$\begin{aligned} \dot{n}_A &= -\alpha n_A n_B + \gamma n_A - \delta n_A^2 \\ \dot{n}_B &= \alpha n_A n_B - \beta n_B \end{aligned}$$

We can simplify this to

$$\begin{aligned} \dot{n}_A &= n_A \left[-\alpha n_B + \gamma - \delta n_A\right] \\ \dot{n}_B &= n_B \left[\alpha n_B - \beta\right] \end{aligned}$$

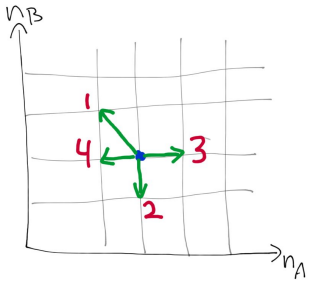
which has a stable fixed point at

$$n_B = \beta/\alpha \quad \text{and} \quad n_A = \frac{\gamma - \beta}{\delta}$$

if $\gamma > \beta$. Let's now assume that this system evolves stochastically. The state of the system is given by $(\mathbf{X} = n_A, n_B)$ and each reaction can induce a state change $\mathbf{\Delta X}$ which are given by

$$\begin{aligned} \mathbf{\Delta X} &= (-1, 1) \\ \mathbf{\Delta X} &= (0, -1) \\ \mathbf{\Delta X} &= (1, 0) \\ \mathbf{\Delta X} &= (-1, 0) \end{aligned}$$

So there are 4 steps the system can take:



The probability rates at which these events occur, however, are state-dependent and assumed to be

$$\begin{aligned} w_1(\mathbf{X}) &= \alpha n_A n_B \\ w_2(\mathbf{X}) &= \beta n_B \\ w_3(\mathbf{X}) &= \gamma n_A \\ w_4(\mathbf{X}) &= \delta n_A \end{aligned}$$

So the total rate, at which the system leaves a state $\mathbf{X}$ is given by the sum

$$\alpha(\mathbf{X}) = n_A(\gamma + \delta) + \beta n_B + \alpha(n_A + n_B).$$

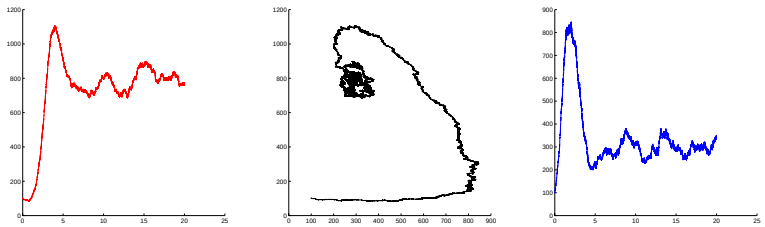
If we want to simulate this process we first initialize the system in a state $\mathbf{X}_0$. The we propagate the system a time T forward, drawn for an exponential distribution

$$p(T; \mathbf{X}) = \alpha(\mathbf{X}) e^{-\alpha(\mathbf{X})T}$$

which is state dependent. After time T we know the system makes on of the 4 steps above with probability

$$p_i(\mathbf{X}) = \frac{w_i(\mathbf{X})}{\sum_j w_j(\mathbf{X})}$$

we pick the step according to this probability and add the corresponding $\Delta\mathbf{X}$ to the current state and compute the next waiting time. And dod this forever. This is called the Gillespie algorithm. For the particular system above, let's look at some trajectories:

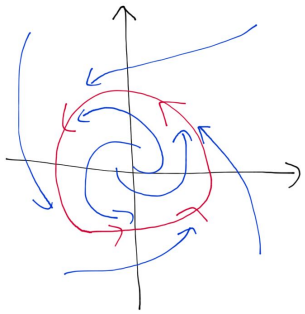


Part II.

Applications

3. Synchronization Phenomena

In nature synchronization happens all the time. In mechanical systems, in biological systems, in epidemiology, basically everywhere. When we talk about synchronization we usually assume that some sort of cyclical dynamics is going on in the units which then synchronize. This means, we need to understand first that usually at the basis of synchronization we are talking about units that if left along perform periodic motion. We have already discussed that this can happen in the form of limit cycles in two-dimensional activator inhibitor systems for example. Now let's assum that we have such an isolated system that performs dynamics in form of a limit cycle in its own state space.

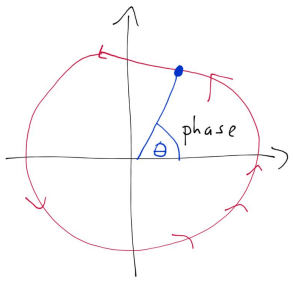


Now effective we can describe the system by a phase θ the position on its limit cycle. Clearly this phase variable θ is periodic

$$\theta(t) = \theta(t + T).$$

and this phase may have different phase velocity at different times along its period

$$\dot{\theta}(t) = \omega(t)$$



In the simplest form, we have a system that has a constant phase velocity

$$\dot{\theta} = \omega$$

3.1. Phase coupled oscillators

Now, clearly if we have many such oscillators, labeled n, each one can have its own internal ω_n and if the oscillators are uncoupled then each one is going to happily move around in a circle at its own velocity

$$\begin{aligned} \dot{\theta}_n &= \omega_n \\ \theta(t) &= \omega_n t \mod 2\pi \end{aligned}$$

From now own, we are not going to write this mod 2π any more and assume that this is always the case in these systems. Now the question is, how can these oscillators synchronize? Clearly thet need to be coupling in some way, to one oscillator can change the phase of another oscillator, or increase or decrease its speed. To generally we would have a situation

$$\dot{\theta}_n = \omega_n + f_n(\theta_1, ...\theta_{n-1}, \theta_{n+1}, ..., \theta_N)$$

if we have a total of N oscillators. The key questions is, what kind of couplings f_n can induce synchrony, and what do we mean by synchrony anyway?

3.1.1. Possible flavors of synchronization

What do we really mean by synchronization? In its simplest form, a multi-oscillator system will evolve into a system in which every unit does exactly the same, i.e.

$$\theta_n(t) = \theta_m(t)$$

for all n, m. This also implies that

$$\dot{\theta}_n = \dot{\theta}_m$$

which means that in the synchronous state, the angular velocities are also identical.

We can, however, also encounter situations in which only the velocities are the same but the individual pairs on oscillators possess a phase different that is constant

$$\theta_n(t) = \theta_m(t) + \alpha_{nm}$$

We will see examples of both.

3.1.2. Two phase coupled oscillators

Let's look at a system of two oscillators that influence each other

$$\begin{aligned} \dot{\theta}_1 &= \omega_1 + f_1(\theta_1, \theta_2) \\ \dot{\theta}_2 &= \omega_2 + f_2(\theta_1, \theta_2) \end{aligned}$$

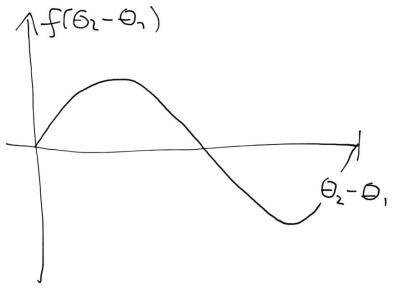
Let's simplify this a bit. Let's assume that the coupling functions are the same so

$$\begin{aligned} \dot{\theta}_1 &= \omega_1 + f(\theta_1, \theta_2) \\ \dot{\theta}_2 &= \omega_2 + f(\theta_2, \theta_1) \end{aligned}$$

Another assumption is that the coupling depends on the phase difference, so

$$\begin{aligned}\dot{\theta}_1 &= \omega_1 + f(\theta_2 - \theta_1) \\ \dot{\theta}_2 &= \omega_2 + f(\theta_1 - \theta_2)\end{aligned}$$

and because the phases are periodic we would like to use a function f that is periodic, too. Now let's assume that $\theta_2(t) > \theta_1(t)$, which means that the second oscillator is ahead. If the function $f(\theta_2 - \theta_1)$ is positive for a positive difference then this will accellerate the first osccilator and decellerate the second, moving them closer.



So let's assume

$$f(x) = \frac{K}{2}\sin(x)$$

so

$$\begin{aligned}\dot{\theta}_1 &= \omega_1 + \frac{K}{2}\sin(\theta_2 - \theta_1) \\ \dot{\theta}_2 &= \omega_2 + \frac{K}{2}\sin(\theta_1 - \theta_2)\end{aligned}$$

where $K > 0$. This is a simple version of the so called Kuramoto model. Let's see what happens to the phase difference

$$x = \theta_2 - \theta_1$$

$$\begin{aligned}\dot{x} &= \mu + K/2\,[\sin(-x) - \sin(x)] \\ &= \delta\omega - K\sin x\end{aligned}$$

where $\delta\omega = \omega_2 - \omega_1$. This dynamical system has a solution on the interval $x \in [0, 2\pi]$ if we can find a solution to

$$\sin x = \frac{\delta\omega}{K}$$

3.1.2.1. $\omega_1 = \omega_2$

Let's first consider this. In this case we have solutions

$$x_1^\star = 0 \quad \text{and} \quad x_2^\star = \pi$$

so that means the oscillators are in phase of anti-phase. Because the derivative of $-\sin x$ is negative at the first fixed point, this one is stable the other is unstable. This means that for any value of the coupling K (arbitrarily small) the oscillators will go into phase synchrony,

$$\theta_1(t) = \theta_2(t)$$

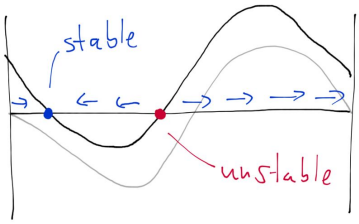
and they will move around in their natural identical frequency $\omega_1 = \omega_2$. So, two identical oscillators will synchronize in the Kuramoto model for any coupling strength K

3.1.2.2. $\omega_1 \neq \omega_2$

This is of course the more interesting case. Let's assume that $\omega_2 > \omega_1$. Looking at the above equation

$$\sin x = \frac{\delta\omega}{K}$$

we see that this has only solutions if $|\delta w/K| \leq 1$. so if either the coupling is sufficiently strong, or the difference in natural frequencies sufficiently small. Let's assume that we are in this regime, the the above equation has solutions where the constant $\delta\omega/K$ intersects the curve $f(x) = \sin(x)$



That happens at two points. One is α the other is $\pi - \alpha$ and

$$\alpha = \sin^{-1} \delta\omega/K$$

That means, the system will evolve into a state

$$\theta_2(t) = \theta_1(t) + \alpha$$

that means the faster oscillator (refering to its natural frequency) is leading the other one by a constant phase. What are their frequencies?

$$\begin{aligned}\dot{\theta}_1 &= \omega_1 + \frac{K}{2}\sin\alpha \\ \theta_2 &= \omega_2 - \frac{K}{2}\sin\alpha\end{aligned}$$

which is

$$\begin{aligned}\dot{\theta}_1 &= \omega_1 + \frac{\omega_2 - \omega_1}{2} = \frac{1}{2}(\omega_1 + \omega_2) = \bar{\omega} \\ \theta_2 &= \omega_2 - \frac{\omega_2 - \omega_1}{2} = \frac{1}{2}(\omega_1 + \omega_2) = \bar{\omega}\end{aligned}$$

so they both compromise and move around at the mean frequency.

We can look at the problem this way. Given to oscillators with ω_1 and ω_2 we need at least a coupling strength

$$K > K_c = \omega_2 - \omega_1$$

such that these two oscillators go into sync.

3.1.3. Many phase coupled oscillators

In most applications we have many oscillators that are coupled, each one interacting with some or all the other ones at a different strength. We can generalize the above approach directly to many oscillators labeled n, each one with its own internal frequency ω_n in which case we would have a

$$\dot{\theta}_n = \omega_n + \frac{1}{N}\sum_m K_{nm}\sin(\theta_m - \theta_n)$$

The entire complexity of what this system can do is hidden in the coupling K_{nm}. And depending on this coupling, lots of different things can happen. Before we discuss these we need to develop ways to measure synchrony in oscillator systems.

3.1.3.1. Measuring the degree of synchrony

The common way of doing this is based on tricks borrowed from complex variables. A complex variable is essentially a 2D vector that has a length r and a phase θ and can be expressed as

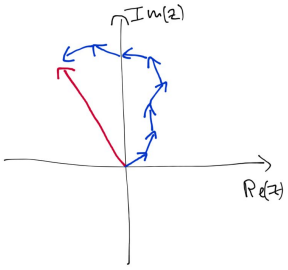
$$z = re^{i\theta} = x + iy$$

Sum's of complex variables are just sums of little vectors:

$$z = \sum_{i=1}^{N} z_i$$

The resulting sum, also has a length and a total phase

$$Z = \sigma e^{i\psi}$$



Now for our system we can assume that every oscillator has a fixed length $r = 1/N$ and its phase continuously changes over time accordint to $\theta_n(t)$. And we can write

$$\sigma(t)e^{i\Psi(t)} = \frac{1}{N}\sum_{n=1}^{N} e^{i\theta_n(t)} = \left\langle e^{i\theta_n(t)} \right\rangle$$

If the system evolves into a state in which $\theta_n(t) = \Omega t + \alpha_n$ then

$$\sigma e^{i\Psi(t)} = e^{i\Omega t}\left\langle e^{i\alpha_n} \right\rangle$$

which means that

$$\Psi(t) = \Omega t + \Psi_0$$

is the overall movement of all oscillators and Ω their synchronized frequency. In this case the prefactor is

$$\sigma = \left|\left\langle e^{i\alpha_n} \right\rangle\right|$$

This means that the constant σ is a measure for the phase coherence in the synchronized state.

3.1.3.2. All to all coupling of identical oscillators

Now let's assume that all N oscillators influence each other by the same amount and have the same frequency

$$\dot{\theta}_n = \omega_0 + \frac{K}{N}\sum_m \sin(\theta_m - \theta_n)$$

Now let's assume that we find a solution in which

$$\theta_n(t) = \omega t + \alpha_n$$

That means that

$$0 = \delta\omega + \frac{K}{N}\sum_m \sin(\alpha_m - \alpha_n)$$

We can see that $\omega = \omega_0$ and $\alpha_n = \alpha_m$ for all oscillators is a solution to the dynamical system. Is it stable? It turns out that this is indeed a stable solution for all initial conditions as long as the oscillators have identical natural frequencies.

3.1.3.3. Some results on more general cases

If we have identical oscillators but variability in the coupling strength K_{nm} one can show that as long as the coupling is symmetric

$$K_{nm} = K_{mn}$$

the dynamics will also always converge to a synchronous state. However, in this case the oscillators have constant phase deviations. Note that the above equation does not imply that all the couplings are the same, only pairwise they are. This means that there can be some spatial heterogeneity, for example the oscillators can be coupled in a network topology. And indeed, in scenarios like that, oscillators synchronize but with a phase distribution.

There are a bunch of more results for the Kuramoto model but they go to deep to discuss them here. Essentially, of one has a distribution of natural frequencies one can expect that synchronization breaks down if the coupling is too weak. If all oscillators are identical then they all synchronize but with a potential distribution of phase differences.

3.2. Pulse-coupled oscillators

The above model is phenomenological and assumes that the force by which individual oscillators influence one another is a smooth function of their phase difference which is largest when a phase-difference of a pair of oscillators is $\pi/2$. In many natural systems oscillators only interact in small periods of time and in form of short pulses. These oscillators are called pulse-coupled oscillators. For example the fire-flies only emit a light pulse which is perceived by the other flies. Another example are neurons which integrate their inputs and then generate an action-potential which is a short pulse that is transported via the neuron's axon and synapses to the receiving neurons which then integrate this input to generate a spike themselves. The simplest model for a neuron is the integrate and fire neuron where the quantity of interest is a membrane potential that follows a dynamics given by

$$\dot{x} = S_0 - \gamma x$$

where S_0 is a constant input and γ models the decay of the membrane potential, which is why these neurons are called leaky integrate and fire neurons. The clue is now if the membrane potential reaches a threshold $x_c = 1$ the neuron fires an action potential and

the membrane potential is immediately reset to the resting potential of $x = 0$. If we start the membrane potential at $t = 0$ at $x(0) = 0$ the above dynamical system can be solved

$$x(t) = \frac{S_0}{\gamma}(1 - e^{-\gamma t})$$

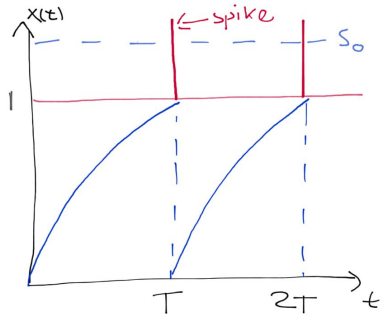
as long as $x(t) < 1$ if $S_0/\gamma > 0$ this solution will hit the critical value $x_c = 1$ when

$$1 = \frac{S_0}{\gamma}(1 - e^{-\gamma T})$$

so at time

$$T = \frac{1}{\gamma} \log \left(\frac{S_0}{S_0 - \gamma} \right)$$

at which time the membrane is reset to zero and the system emits a pulse. Such an oscillator left alone will emit pulses continuously at a period T. The curve of the membrane potential looks like this:



and we can write it

$$x(t) = \frac{1 - e^{-\gamma t}}{1 - e^{-\gamma T}}$$

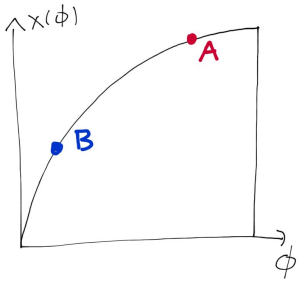
We can express time in units T in which the equation becomes

$$x(\theta) = \frac{1 - e^{-\gamma \theta}}{1 - e^{-\gamma}}$$

Now we have to specify what and oscillators pulse does to another oscillator. Lets first discuss a system of two such oscillators.

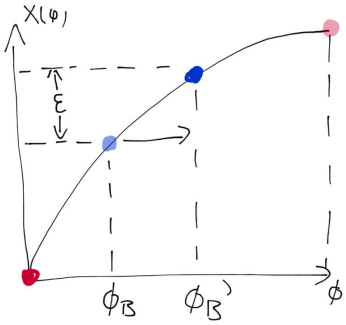
3.2.1. Two pulse-coupled oscillators

Let's look at two oscillators A and B each one being in some phase ϕ. So let's call these state ϕ_A and ϕ_B. Let's assume that A is closer to it's firing time, to $\phi_A > \phi_B$. We can visualize this by drawing dots on on the curve in the x, ϕ plane:



If A reaches the firing threshold it's phase is reset to 0 and the membrane potential of B gets boosted by a value $\epsilon > 0$. This boost in membrane potential x is equivalent to an advance in the phase of B

$$\phi_B \to \phi'_B.$$



How do we compute the new phase ϕ'_B? We have

$$x = f(\phi) = \frac{1 - e^{-\gamma\phi}}{1 - e^{-\gamma}}.$$

so the new phase is

$$\phi'_B = g(x + \epsilon)$$

where the function $g(x)$ is the inverse of $f(\phi)$. We can see that under some very general assumptions this kind of system can synchronize oscillators of the pulse of a firing oscillator advances the other oscillator sufficiently. However that second oscillator when it fires will advance the first and in principle they could be racing one another forever. This is why the function $f(\phi)$ has to be concave down. So that the first oscillator advances the second one more that vice versa. One the both fire together, the will stay in synchony.

3.2.2. Many pulse-coupled oscillators

Now let's look at N such identical oscillators that are all coupled to one-another. One can show that if the oscillators are identical then they will eventually all synchronize. An intuitive explanation is that they can never desynchronize once they are in synchrony and that eventually clusters of synchronized oscillators emerge that will continue to form larger clusters.

3.3. Diffusively coupled oscillators and patterns

Let's get back to phase coupled oscillators and imagine that each oscillator is on a site in a 1D lattice: Recall that the dynamic equation is given by

$$\dot{\theta}_n = \omega_n + \frac{1}{N}\sum K_{nm}\sin(\theta_m - \theta_n).$$

Now let's assume that each oscillator only interacts with its neighbors and that these interactions are all of the same strength, in this case we have

$$\theta_n = \omega_n + \frac{K}{2}\sum_{m\in G_n}\sin(\theta_m - \theta_n)$$

where G_n is the set of 2 neighbors of n. Explicity we can write

$$\dot{\theta}_n = \omega_n + K\left[\frac{\sin(\theta_{n+1} - \theta_n) + \sin(\theta_{n-1} - \theta_n)}{2}\right]$$

Now let's assume we have very many of those guys and that as we go from grid to the nect grid the phase doesn't change so much. In this case we can use the series expansion of the sin

$$\sin(x) = x - \frac{1}{3}x^3....$$

so the equation is approximately governed by

$$\dot{\theta}_n = \omega_n + K\left[\frac{\theta_{n+1} - 2\theta_n + \theta_{n-1}}{2}\right]$$

If we now identify n with a spatial coordinate $x_n = n\Delta x$ and let $K = D/\Delta x$ this becomes

$$\partial_t\theta(x,t) = \omega(x) + D\partial_x^2\theta(x,t)$$

where we interpret $\theta(x,t)$ as a continuous array of oscillators that are now diffusively coupled. This means that the phase "diffuses" to the neighboring locations. The function $\omega(x)$ determines the local natural frequency density of oscilators.

This system also develops synchrony and the spatial pattern is determined by the strength of diffusion. Depending on the variability of $\omega(x)$ we can see that the oscillator field converges to oscillatory behavior with a uniform frequency but with a phasic decoherence, just like in the original phase coupled system.

3.4. Coupled Limit cycle systems

Let's now investigate a system that helped us motivate the phase model as well as the pulse coupled system. We discussed that systems, e.g. activator inhibitor systems, that naturally yield limit cycle behavior are candiates for being modelled just by focusing on

the phase. Let's see what happens if we model a full activator inhibitor system in the sense that we explicitely model each oscillator by

$$\begin{aligned} \dot{u} &= f(u,w) \\ w &= g(u,w) \end{aligned}$$

where the variables u, w are the state variables of one oscillator. A simple system that generates limit cycles if given by

$$\begin{aligned} \dot{\theta} &= \Omega \\ \dot{r} &= \lambda r(1-r) \end{aligned}$$

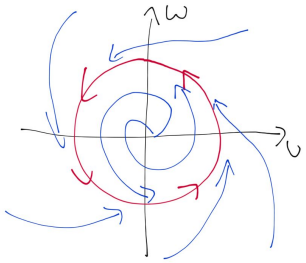
where $r = \sqrt{u^2+v^2}$ and $\theta = \tan^{-1} w/u$, or $u = r\cos\theta$ and $w = r\sin\theta$. We can relate both coordinate systems by

$$\begin{aligned} \dot{u} &= \dot{r}\cos\theta - r\dot{\theta}\sin\theta \\ \dot{w} &= \dot{r}\sin\theta + r\dot{\theta}\cos\theta \end{aligned}$$

so

$$\begin{aligned} \dot{u} &= \lambda(1-\sqrt{u^2+y^2})u - \Omega w \\ \dot{w} &= \lambda(1-\sqrt{u^2+y^2})w + \Omega u \end{aligned}$$

This means that u is the activator and w is the inhibitor and trajectories go to a limit cyctle $\dot{\theta} = \Omega$ and $r = 1$.



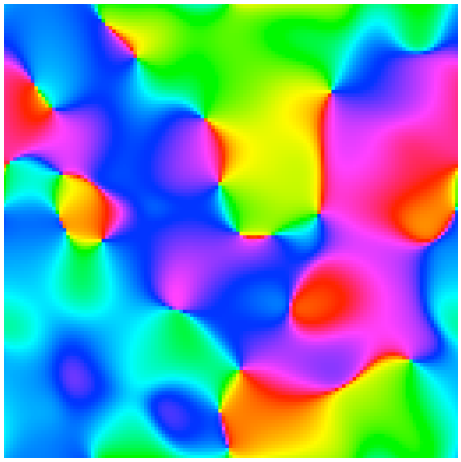
Now lets assume that both, u and w are concentrations of a chemical or a species on some site n. and both concentrations and diffusively change the concentration on neighboring sites so

$$\begin{aligned} \dot{u} &= \lambda(1-\sqrt{u_n^2+w_n^2})u_n - \Omega w_n + \frac{D_u}{2}\left[u_{n+1}+u_{n-1}-2u_n\right] \\ \dot{w}_n &= \lambda(1-\sqrt{u_n^2+w_n^2})w_n + \Omega u_n + \frac{D_w}{2}\left[w_{n+1}+w_{n-1}-2w_n\right] \end{aligned}$$

which means that if the concentration w_{n+1} and w_n are very different the last term is going to make them closer. Effectively coupling the dynamics at each site. If we now consider the spatially continuous model this reads

$$\begin{aligned} \partial_t \dot{u}(x,t) &= \lambda(1-\sqrt{u^2(x,t)+w^2(x,t)})u(x,t) - \Omega w(x,t) + D_u\partial_x^2 u(x,t) \\ \partial_t \dot{w}(x,t) &= \lambda(1-\sqrt{u^2(x,t)+w^2(x,t)})w(x,t) + \Omega u(x,t) + D_w\partial_x^2 w(x,t) \end{aligned}$$

Patterns generated by this model also exhibit synchronization but what interesting is the emergence of locations where the phase θ exhibits a singularity. These are pinwheels. Locations in the fields where oscilators can't decide what to do. In the long time limit these singularities can annihilate by moving towards one another. This only can happen if two of these singularities have the opposite charge.

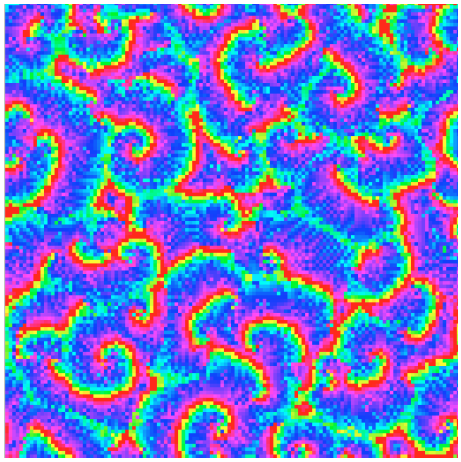


3.5. Spatially distributed pulse coupled oscillators

The above system, that produces these beautiful pinwheel patterns motivates us to ask what kind of patterns can emerge in pulse coupled oscillators in which each oscillator is governed by the equation

$$\dot{x} = S_0 - \gamma x$$

when below the threshold $x_{th} = 1$ and in the regime $S_0/\gamma > 1$. In this regime every oscillator fires at a constant period. Now lets assume that on a lattice each oscillator receives input only from its metric neighbors in the lattice. This system generates non-equilibrium patterns that consists either of target or spiral waves. Here's an example:



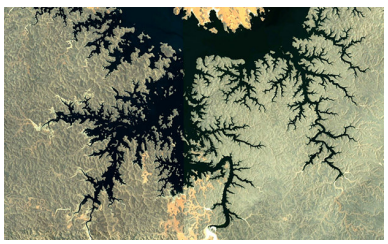
Why is this happening. The spirals are rotating regions of repetative activation. This happens because in the pulse coupled scenario, even though the coupling is symmetric between oscillators in terms of strength, the pulse occur only in one direction which introduces a temporal asymmetry. These spiral patterns are very generic in excitable systems in which indiviudal units can be pushed above threshold and become active and feed-forward activation throughout the system. We will talk more about these patterns in the nect chapter.

4. Pattern Formation

The last two examples of spatially distributed oscillators serve as a good intro to one of the most important aspects in complex systems: pattern formation. By pattern formation we mean that certain systems have the ability to self-organize into spatially structured states from intitially unstructured or spatially homogenous states. This behavior takes place all over the place, in physics in chemistry in biology in social science etc. Here we will discuss the basic ingredient that are necessary for these processes to occur. One of the patterns that we see in any desert or any beach are ripples in the sand:

One can understand the basic ingredients for pattern formation in this system. Laminar flows of wind move accross the sand and move individual sand grains with it. Slight pertubations in the surface have a higher likelihood of collecting these grains increasing the magnitude of the pertubation and thus increasing the likelihood of catching more sand. This is a positive feedback effect which is necessary in almost all pattern forming systems. This example also shows that one needs to put energy into pattern forming systems, in this case wind. Right behind the pertubation a dip forms, as the probability of grains collecting there becomes lower. Of course the increasing hills cannot increase forever and will be eroded at the top. This is the negative feedback that is also required in pattern formation.

The same thing happens when river beds are formed. There's a continuous precipitation that falls onto the land and the water streams it forms follows gravity downhill. Small streams carve a path and the following water will preferentially follow those paths, too digging deeper into the soil. Of course this can't go on foverever. The process can produce valleys that collapse and new valleys will form:

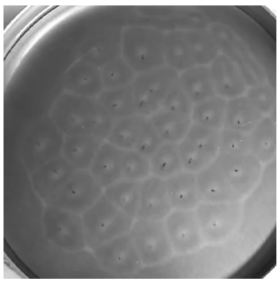


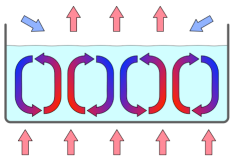
Let's discuss a few examples that all show these type of ingredient.

4.1. Examples of pattern forming in nature

4.1.1. Physics

The most known example of spontaneous pattern formation in physics is probably Benard convection. This happens when a liquid is heated from below at such a high rate that the heat cannot dissipate through the system fast enough. In this case convetion rolls or cells emerge that transport the heat to the cooler regions of the liquid where heat is given off. The cooled liquid then gets pushed to the bottom to be reheated:





This experiment can be easily done by heating oil on the stove and adding metal dust to the oil. Again we have to add energy to the system and once certain parts of the liquid start moving upwards this process will be accelerated and other regions have to move downward to keep the liquid in place. Images of the suns surface show granules which are exactly these kind of convetion pockets.

4.1.2. Chemistry

There's an abundance of chemical reactions that, if they occur in a not-well-stirred scenario, have the potential of generating patterns. One such reaction, the most famous one, is the Belousov–Zhabotinsky reaction. This reaction uses essentially five reactions,

	Reaction	Rate
(O1)	$A + Y \to X + P$	$k_3 = k_{R3}[H^+]^2 AY$
(O2)	$X + Y \to 2P$	$k_2 = k_{R2}[H^+]XY$
(O3)	$A + X \to 2X + 2Z$	$k_5 = k_{R5}[H^+]AX$
(O4)	$2X \to A + P$	$k_4 = k_{R4}X^2$
(O5)	$B + Z \to \frac{1}{2} f Y$	$k_0 BZ$

in which particular chemicals in it start oscillating in their concentration. If the reaction takes place in a petri dish, spiral and target wave form. This is because in its essence this reaction is an activator inhibitor system in which an autocatalytic reaction increases the abundance of a chemical which in turn increases the abundance of a chemical that turns this reaction off. Here's a snapshot of a pattern that the BZ reaction can generate:

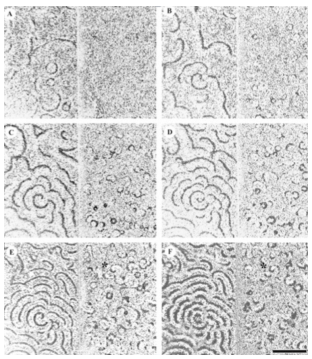
This looks very similar to the patterns we generated with the spatially distributed pulse-coupled oscillators. And in fact the mechanisms in both systems are very similar. We will come back to this later.

4.1.3. Biology

Biology is full of pattern forming systems, so we are going to look at numerous examples.

4.1.3.1. cAMP signaling in Dictyostelium discoideum

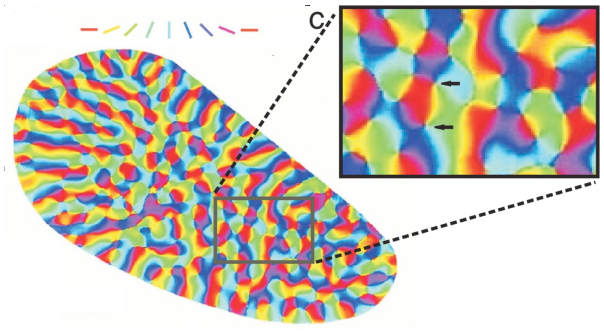
Dictyostelium discoideum is an interesting organism. It's a single cell amoeba that, when nutrient run low excretes a signalling molecule called cAMP which other individuals in the neighborhood respond to in two ways. 1.) they move up the gradient towards the source, getting closer to the orignator of the signal and, 2.) the start excreting cAMP, too. This is positive feedback. Eventually this leads to the aggregation of thousands of these organisms into a multicellualr organism, a slug like creature that crawls away and develops into a differentiated mold that forms a fruiting body and a stem, spores that get carried away by wind. If one measures the concentration of cAMP one sees spiral waves, just like in the BZ reaction and the pulse coupled oscillators:



4.1.3.2. Orientation Maps in the visual cortex of primates

An interesting pattern is seen on the visual cortex of primates and other higher mammals that have binocular vision. In the visual cortex signals that come in from the receptors

in the eyes are processed. It turns out that the visual cortex has a bunch of cells that respond best to stimuli of a particular orientation, for instance contrast contours that are horizontal or vertical etc. If one records response strengths from neurons in the visual cortex one can color code the locations in the visual cortex according to the prefered stimulus orientation and this is what one gets:



We see a pattern that is reminiscent of the pinwheel pattern we saw in the phase coupled oscillator lattice. We see singularties, the pinwheels where all orientation meet. The cool part about thes visual maps is that animals aren't born with them. These maps develop after birth and in response to visual stimuli.

4.1.3.3. Patterns on sea shell

Here's a picture of natural patterns of different snail shells. These shells grow slowly over time by continuous deposits of material and the distribution of types of material and interactions between regions in the organism that make the hard material generate different patterns. This is an example of a growth process that yields complex patterns. Something we will come back to.

This is an interesting example because not only the patterns on the shels are interesting but the patterns of the shells themselves which appear to be following some very basic formation rules. In fact, a whole book was written by one of the pioneers of pattern formation in developmental biology: Hans Meinhardt. He developed a model for pattern formation which we will discuss in detail.

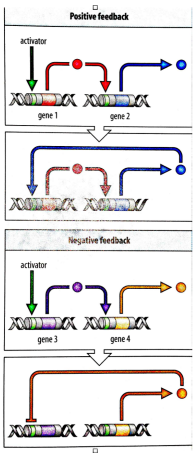
4.1.3.4. Patterns in animal fur

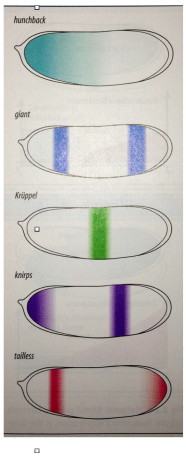
And of course, when we think of patterns in organism we have to mention animal fur:

This picture shows different types of animal fur. Some patterns are spotty, some consist of stripes of different wavelength. It turns out that will very few dynamic ingredient one can devise a model that is capable of generating all these patterns at different parameter values.

4.1.3.5. Biological Morphogenesis

The above pattern in animals are a special case of something a lot more fundamental. The problem of how multicelluar organisms with a great diversity of function and cell morphology can develop from a single, homogeneous fertilized egg. First, all cells in a multicellular organism have the same genome despite the variety in shape and function. This can be the case because these cells differ in what genes are expressed, i.e. active or repressed, i.e. inactive. On the way towards an adult organsism, cells differentiate by a sequential switching off an on of genes. This mechanism is responsible for spontaneouly introducing differences in embryos. For example in the drosophila melanogaster (fruitfly) embryo, different genes are expressed in different regions. The combination of expression levels introduces different cell fates for the cells in specific locations which eventually governs the development of a full blown fly:



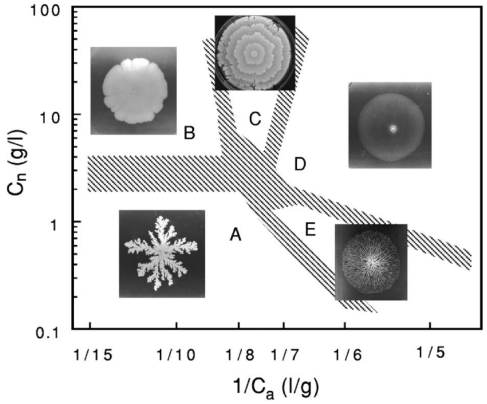


This process is very autonomous. If one disturbs gene expression in certain locations one can generate interesting effects, for instance a fruit fly that has a full functioing middle segment with additional wings:

Or mice with additional digits:

4.1.3.6. Growth of bacterial colonies

Here's another interesting example of a growing colony of the bacterium bacillus subtilis in a petri dish of nutrients. The control parameters are the physical properties of the agar in which the bacteria can move around and the concentration of nutrients. As a function of motility and nutrient concentration different growth patterns emerge. We will discuss these types of patterns later when we talk about diffusion limited aggregation and growth processes:



4.1.4. Alan Turing

The guy who first thought and published about the origins of spontaneous pattern formation was Alan Turing, who published a seminal paper called "The chemical basis of morphogenesis". In this paper, published only shortly before his death, Turing argued that an abundance of patterns observed in nature, many of the ones mentioned above, can be generated by the interaction of three different ingredients.

1. Activation
2. Inhibition
3. Diffusion

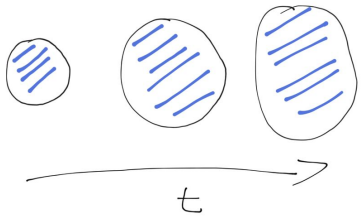
The basic idea being that essentially two types of agents (e.g. molecules, animals or other dynamical quantities) interact. An activator that does autocatalysis, which means this activator generates more of itself. Also, the activator triggers the generation of an inhibitor. The action of the inhibitor decreases the abundance of the activator. We have seen examples of such dynamical systems. The clue is that both, activator and inhibitor can move in space diffusively. So imagine you have a small concentration of an activator that subsequently generates more of itself and diffuses which may trigger a travelling wave of activation. However, the inhibitor is also generated and if that inhibitor diffuses faster it will eventually stop the activator. Activator can diffuse beyond the wall of inhibitor and generate a new activator nucleation and the process repeats causing a systems of stripes or rings to emerge. We will discuss how generic this process is by looking at numerous representative examples.

4.2. Transient and stable patterns

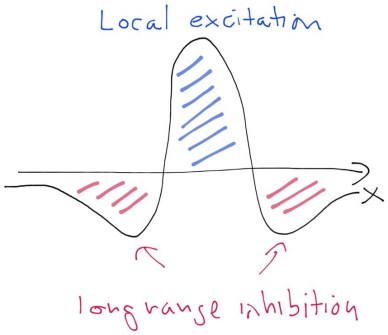
We've seen a couple of examples of patterns so far. It's reasonable to make the distinction between transient patterns that come and go, and those that stabilize. For instance, in one if we have a pattern that can be described by a fisher equation

$$\partial_t u = \lambda u(1-u) + D\partial_x^2 u$$

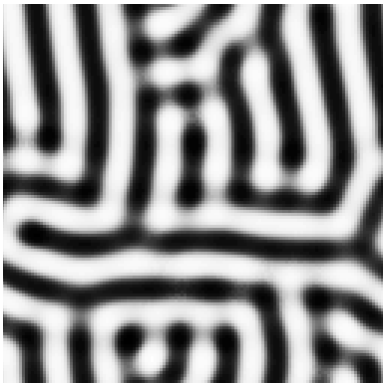
then for the appropriate initial condition we can observe a propagating wave that will spread throughout the entire system.



This is an example of a transient pattern.

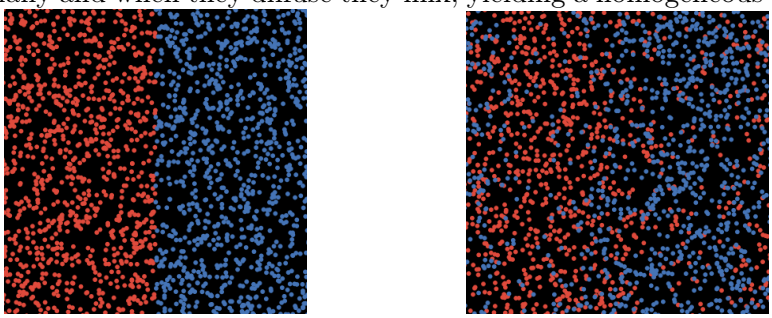


In one of the homework assignments we looked at a system in which the activity at a location was increased by the activity in a smaller radius and decreased by the activity of element in a larger radius. This local excitation and long range inhibition was able to produce stable stripelike patterns.



In that assignment, though, this mechanism was imposed and we need non-local interactions. In many physical systems, e.g. when particles can only move, e.g. diffuse,

this non-local interaction doesn't work. A question is now whether diffusion mechanism can effectively yield this type of local excitation and long range inhibition. Intuitively we would expect that this isn't possible because diffusion usually enhances homogeneous patterns, i.e. distributions which are initially non-homogeneous go to a more homogeneous state. E.g. in the picture below, we've got diffusive particles of two types intitally separated spatially and when they diffuse they mix, yielding a homogeneous distribution.



4.2.1. Ad-hoc stable patterns

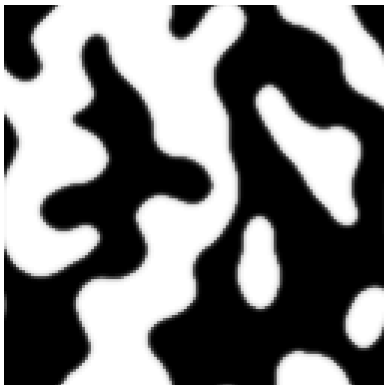
Let's try to construct a system that can potentially generate spatial patterns. Let's look at a dynamical system

$$\dot{u} = u - u^3$$

this is a dynamical system that has three fixedpoints, the unstable fixed point $u = 0$ and stable fixed points at $u = \pm 1$. Let's be experimental and look at this system, but spatially extended.

$$\partial_t u = u - u^3 + \partial_x^2 u.$$

This is what this system generates:



A nice pattern of regions that are either black or white ($u = \pm 1$). However, this is not such a surprise because the local dynamics already has two stable solutions and the diffusion only amplifies the initial condition. In fact these patterns change continuously and will eventually go into a homogeneous state. More importantly, the question is whether diffusion can introduce the emergence of a pattern in a system which otherwise would be homogeneous.

4.3. The Turing mechanism

It turns out that diffusion can also generate pattern in a system that without diffusion would be going to a stable homogeneous state. The idea is the following. Essentially many systems consist of an activator and an inhibitor. Let's for instance look at

$$\begin{aligned} \partial_t u &= u^2 w - u \\ \partial_t w &= \beta - u^2 w \end{aligned}$$

In this dynamical system u is an autocatalytic chemical that is activated by itself and ω and degrades spontaneously. When it increases it inhibits w by removing it. This system has a stationary solution

$$u^\star = \beta \quad \text{and} \quad w^\star = 1/\beta.$$

Let's see if this state is stable we need to compute the Jakobian

$$A = \left(\begin{array}{cc} f_u & f_v \\ g_u & g_v \end{array} \right) \Big|_{u^\star, v^\star} = \left(\begin{array}{cc} 1 & \beta^2 \\ -2 & -\beta^2 \end{array} \right)$$

And remember that this is stable if the trace of A is negative and the determinant is positive

$$1 - \beta^2 < 0 \quad \text{and} \quad \beta^2 > 0$$

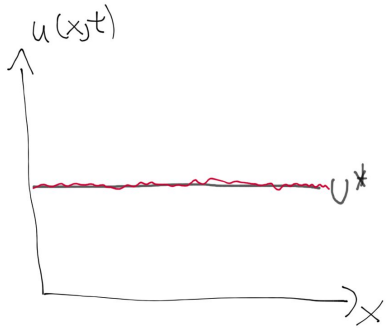
This means, the systems stationary state is stable if $\beta > 1$. Now let's imageing that

$$u = u(x,t) \quad \text{and} \quad v = v(x,t)$$

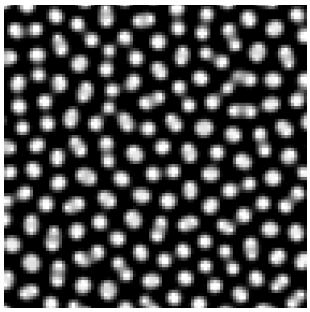
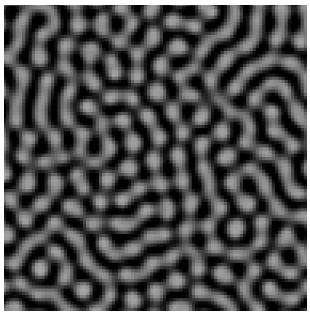
and that both chemicals can also diffuse with different diffusion coeffients. In this case

$$\begin{aligned} \partial_t u &= u^2 w - u + D_u \partial_x^2 u \\ \partial_t w &= \beta - u^2 w + D_w \partial_x w. \end{aligned}$$

Now if we distribute the chemicals uniformly then the partial derivatives with respect to the spatial coordinate vanish and the system behaves like the local system. The question is, what happens to slight pertubations from the stationary state.



. Do they get amplified or are they damped down. Let's see what can happen in the above system when we simulate it. We pick the parameters $\beta = 1.5$ and $D_w/D_u = 4$. What we see if we solve the above system numerically is that everything moves to the globally homogeneous state. However if we increase the diffusion of the inhibitor $D_w = 16$ the homogeneous state beomes unstable and a pattern consisting of spots or stripes emerges.

This is a general effect: If the diffusion of inhibition happens a lot faster than that of the activator, the system undergoes a Turing instability and patterns emerge which are called turing patterns. The come in different flavors some of which we will discuss soon. How can we understand this?

It turns out that with a little math we can and we can in fact derive a set of 4 equations that can tell us for what parameters a Touring instability will occur.

4.3.1. Another Example

Let's look at the following system

$$\begin{aligned} \partial_t u &= \frac{u^2}{w} - \mu u + \rho + \partial_x^2 u \\ \partial_t w &= u^2 - \lambda w + \sigma \partial_x^2 w \end{aligned}$$

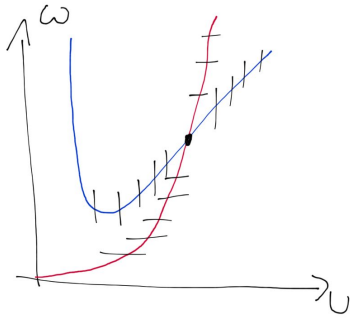
Here u is again an autocatalytic quantity that increases by positive self-coupling. However it also generates w which if it increases decreases the autocatalytic mechanism. Thus w is the inhibitor. Again the local system has one stable fixed point which is where the nullclines

$$w = \frac{u^2}{\mu u - \rho}$$

and

$$w = \frac{1}{\lambda} u^2$$

meet.



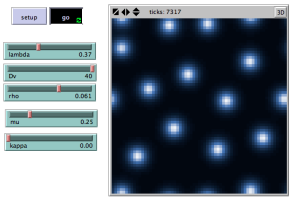
The fixed point is given by

$$u^\star = \frac{\rho + \lambda}{\mu} \quad \text{and} \quad w^\star = \frac{(\rho + \lambda)^2}{\lambda \mu^2}$$

We also observe in this sytem that when

$$\sigma < \sigma_c$$

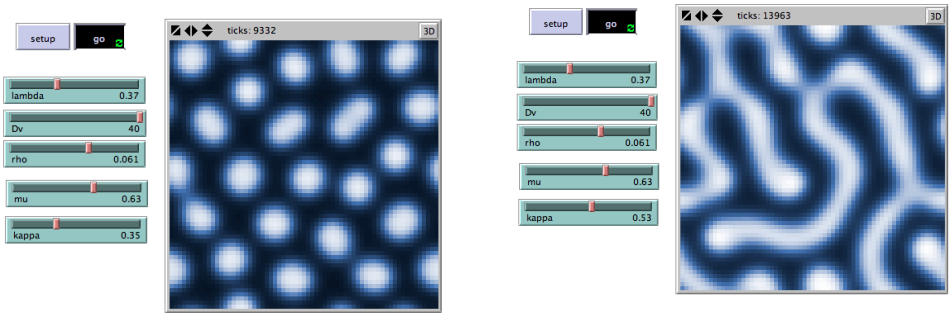
then the sytem goes into a homogeneous state, but when $\sigma > \sigma_c$ a Turing instability occurs and spots form.

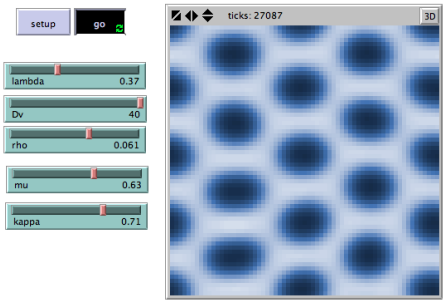


If we modify this dynamical system a little bit by replacing

$$u^2 \to \frac{u^2}{1 + \kappa u^2}$$

which means that the autocalytic process saturates and effect introduced by the new parameter κ the system is able to produce a variety of patters out of the turing instability.





4.3.2. Conditions for a Touring instability.

We can play with these system and will always find that if the inhibitor diffuses faster a turing instability will occur. But it's of course also important to know when. So let's see how far we can get by looking all systems of the form

$$\begin{aligned} \partial_t u &= f(u,w) + D_u \partial_x^2 u \\ \partial_t w &= g(u,w) + D_w \partial_x^2 u \end{aligned}$$

which we can also write as

$$\begin{aligned} \partial_t u &= f(u,w) + \partial_x^2 u \\ \partial_t w &= g(u,w) + \sigma \partial_x^2 u \end{aligned}$$

Now we assume one stable fixed point of the spatially homogeneous system:

$$f(u^\star, w^\star) = g(u^\star, w^\star) = 0$$

We have to compute the Jakobian

$$A = \left(\begin{array}{cc} f_u & f_v \\ g_u & g_v \end{array} \right) \Bigg|_{u^\star, v^\star}$$

and the fixed point is stable if

$$s = \mathrm{Tr} A < 0 \quad \Delta = \det A > 0$$

which meanst that

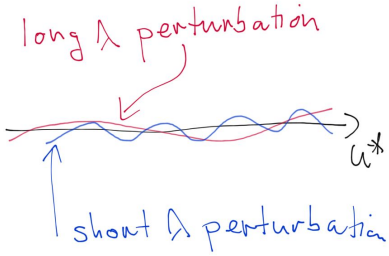
$$f_u + g_v < 0$$

and

$$f_u g_v - f_v g_u > 0$$

These are two conditions on the dynamical system for stability. Now look at spatial system in the vicinity of uniform solution, meaning that we perturb the system with a spatial pertubation:

$$\begin{aligned} u(x,t) &= u^\star + \delta u(x,t) \\ w(x,t) &= w^\star + \delta w(x,t) \end{aligned}$$



Let's pack the two variables into a vector

$$\mathbf{U} = (\delta u, \delta w)$$

Then this vector evolves according to

$$\partial_t \mathbf{U} = A\mathbf{U} + D\partial_x^2 \mathbf{U}$$

with a matrix

$$D = \begin{pmatrix} 1 & 0 \\ 0 & \sigma \end{pmatrix}$$

The question is what happens to the pertubation $\mathbf{U}$, is it going to grow or vanish. Let's make the following ansatz for a solution

$$\mathbf{U}(x,t) = \mathbf{U}_0 e^{\lambda t} \cos(kx)$$

If this is a solution to the equation and if $\mathrm{Re}\lambda > 0$ this solution will grow expontieally and the pertubation is unstable. If on the other hand $\mathrm{Re}\lambda < 0$ the pertubation will go away. We are looking at a pertubation that's a cosine wave with wavelength $2\pi/k$. We are doing this so we can see if the behavior of pertubations depends on the nature of the pertubation. Plugging this into the above equation we get

$$\lambda \mathbf{U} = \left[A - Dk^2\right] \mathbf{U}$$

so we need to solve this eigenvalue problem and determine the sign of the eigenvalues. We have to understand the eigenvalues of the matrix $B = A - Dk^2$. which is given by

$$\begin{pmatrix} f_u - k^2 & f_v \\ g_u & g_v - \sigma k^2 \end{pmatrix} = B$$

We know that the nature of the eigenvalues are determined by the trace and determinant of the matrix B so the trace is given by

$$(f_u + g_v) - k^2(1+\sigma) = \mathrm{Tr}B$$

and the determinant by

$$\begin{aligned} \left(k^2 - f_u\right)\left(\sigma k^2 - g_v\right) - f_v g_u &= \det B \\ \sigma k^4 - (g_v + \sigma f_u)k^2 + f_u g_v - f_v g_u &= \det B \\ \sigma k^4 - (g_v + \sigma f_u)k^2 + \det A &= \det B \end{aligned}$$

The steady state is unstable either if

$$\mathrm{Tr} B > 0$$

or if

$$\det B < 0$$

The trace, however is always negative, because $TrA = f_u + g_v < 0$. So we have to find conditions when the determinant becomes negative. Since $\det A > 0$ the only way for the determinant to become negative is:

$$g_v + \sigma f_u > 0$$

If, for example

$$g_v < 0$$

at the fixed point then $f_u > 0$ then the requirement is

$$\sigma > 1$$

which means the inhibitor needs to diffuse faster. Let's set

$$\begin{aligned} p &= (g_v + \sigma f_u) > 0 \\ q &= \det A > 0 \end{aligned}$$

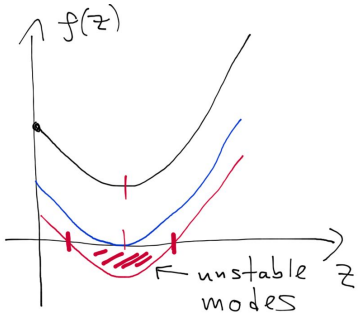
and $k^2 = z$. Then we can write the above equation as

$$\sigma z^2 - pz + q = h(z)$$

and see what it can look like. We have to see where this guy becomes negative. The function is a parabola that has an offset q and a minimum at

$$h_{min} = \frac{p}{2\sigma} = \frac{g_v + \sigma f_u}{2\sigma}$$

and it looks like this:



If we change the parameters we can make this parabola become negativ at a certain value $z_c = k_c^2$ which means that pertubation with this wavelength become unstable. The condition for this happening is when $h(z) = 0$ and this happens when

$$p^2/q \quad > \quad \sigma$$

or

$$\frac{(g_v + \sigma f_u)^2}{4(f_u g_v - f_v g_u)} > \sigma$$

which we can write has

$$(g_v + \sigma f_u)^2 > 4\sigma(f_u g_v - f_v g_u)$$

4.3.2.1. In summary

So that means we have a bunch of conditions:

$$f_u + g_v \quad < \quad 0$$

$$f_u g_v - f_v g_u > 0$$

$$g_v + \sigma f_u > 0$$

$$(g_v + \sigma f_u)^2 - 4\sigma(f_u g_v - f_v g_u) > 0$$

If these conditions are met, then the homogeneous state is unstable and patterns emerge.

4.3.3. Example

Let's look at our example

$$\begin{aligned} \partial_t u &= u^2 w - u + \partial_x^2 u \\ \partial_t w &= \beta - u^2 w + \sigma \partial_x^2 w \end{aligned}$$

We have

$$A = \begin{pmatrix} 2uw - 1 & u^2 \\ -2uw & -u^2 \end{pmatrix} = \begin{pmatrix} 1 & \beta^2 \\ -2 & -\beta^2 \end{pmatrix}$$

and

$$\begin{aligned} f_u + g_v &= 2uw - 1 - u^2 \\ &= 1 - \beta^2 \end{aligned}$$

so the first condition implies $\beta > 1$. The determinant is always positive.

$$\Delta = -\beta^2 + 2\beta^2 = \beta^2 > 0$$

The third condition reads

$$\sigma > \beta^2$$

and finally the fourth condition is

$$\left(\sigma - \beta^2\right)^2 \quad > \quad 4\sigma\beta^2$$

Or

$$\sigma > b^2\left(3 + 2\sqrt{2}\right)$$

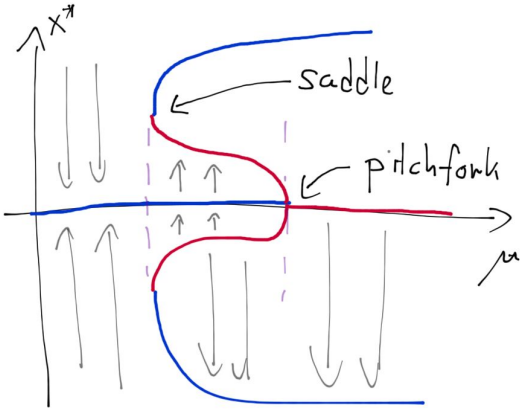
5. Critical Phenomena

Typically the concept of critical phenomena is associated with physical systems that undergo phase transitions at particular temperatures and or pressures. For instance from vapor to liquid to solid. We know that "stuff" consists of molecules and its interesting that for instance when one cools a vapor in which molecules bounce arround randomly suddenly make a transition into a liquid state where there are closer together but still move abount randomly. This transition is a macroscopic phenomenon driven by changes in the way the microspoic things interact. Thinking about it this way we can also apply it to all sorts of systems for example social systems, or biological systems.

Critical phenomena are related to the concept of bifurcations that we discussed when we discussed dynamical systems. Recall the following example

$$\dot{x} = x(\mu - x^2)$$

For any initial condition and $\mu < 0$ the system will aproach the stationary point $x = 0$. Ir we increase μ to a value $\mu > \mu_c = 0$ the system will either go into the one of the states $x = \pm\sqrt{\mu}$. We can think of this system having undergone a critical transition. Remember there was also the idea or hysteresis. IF we have a bifurcation diagram like this:



then that means a system will remain in one state until a critical μ is reached and then will jump discontinously to another state. This is a bit different that a smooth transitions. Also moving the parameter back, doesn't make the system go back into to original state. Although we can observe this sort of behavior in a simple dynamical system, the characteristic feature is also present in more complex critical phenomena.

Likewise, when we talked about pattern forming systems, we had for instance a critical value for the ratio of diffusion coefficients beyond which patterns emerged from a desta-

bilized homogeneous state. This is all good, but this are low dimensional system. The key feature about what is considered critical phenomena are macroscopic structures from microscopic largely independent units that also may have a strong random component.

5.1. Example: An epidemic

Let's discuss the following system first. Let's first look at what is known as the SIS model in epidemiology, We assume that we have a population with a fraction of infecteds y and a fraction of susceptivles x and we assume two reactions occur

$$S + I \xrightarrow{\alpha} 2I$$

$$I \xrightarrow{\beta} S$$

An ODE system for this is given by

$$\begin{aligned} \dot{x} &= -\alpha xy + \beta y \\ \dot{y} &= \alpha xy - \beta y \end{aligned}$$

which we can write as a 1d system

$$\begin{aligned} \dot{y} &= \alpha y(1-y) - \beta y \\ \dot{y} &= \alpha y\left(1 - \frac{\beta}{\alpha} - y\right) \end{aligned}$$

This system always has the disease free state $y = 0$ as a fixed point and also, if $\beta/\alpha < 1$ also the fixed point

$$y = 1 - \frac{\beta}{\alpha}$$

This means if we consider

$$R_0 = \frac{\alpha}{\beta}$$

then if

$$R_0 > R_0^c = 1$$

the system has infecteds and no infecteds if $R_0 < 1$. The question is now whether this is observed in a situation in which we are actually dealing with very many "agents" that act randomly like molecules in the water. This is best done in an agent based simulations. Imagine we have a grid in which grid points

$$u_{ij}$$

can have two states S or I and we randomly update each grid point according to the following rules

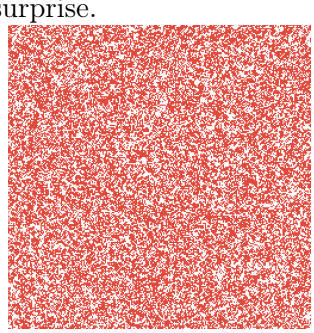
1. If $u_{ij} = I$ then with probability β that gridpoint will go to state S

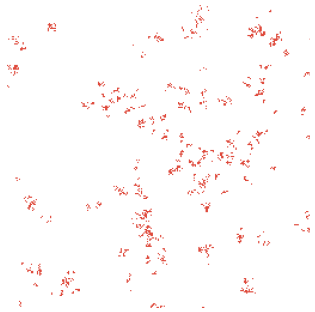
2. If $u_{ij} = S$ the site picks a random neighbor and if that random neighbor is I then the site goes to I with probability α.

If we run this algorithm with netlogo or matlabl we will see that for values $\beta, \alpha \ll 1$ that this system has a critical phase transition as expected, for

$$\alpha > \beta$$

we see a persistent endemic concentration of infecteds and otherwise the disease dies out. This is not a surprise.





5.1.1. SIR Dynamics

A slight variation of the above model consideres three different states for individuals to be in, suscptible, infected and recovered and the following 2 reactions

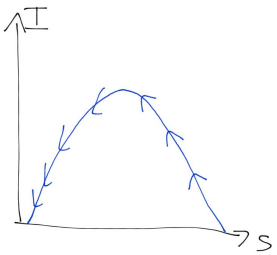
$$S + I \xrightarrow{\alpha} 2I$$

$$I \xrightarrow{\beta} R$$

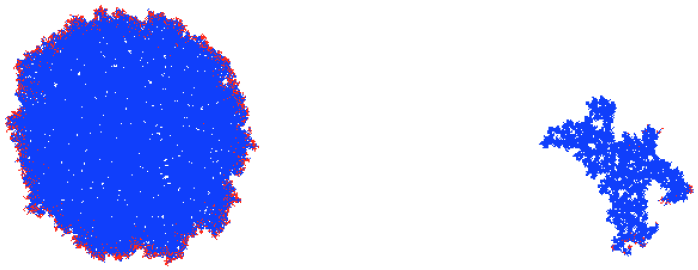
In this case, as we discussed earlier the dynamical system that describes this is given by

$$\begin{aligned} \dot{x} &= -\alpha xy \\ \dot{y} &= \alpha xy - \beta y \end{aligned}$$

In phase space the dynamics looks like this:



IF we start with $y(0) = y_0 \ll 1$ and $x(0) = 1 - y_0$ then if $\alpha > \beta$ a large epidemic will occur. Otherwise the initially small fraction of infected will die out. Let's see how this looks in a random lattice system:



We see that if we vay the ratio α/β we can determine the overall fraction of the number of infecteds after an epidemic. We also see that if we approach a critical value, this number drops off sharply. It is, however, more challending to determine the exact value when this happen. Again, this is not so interesting as such because all that changes is the overall magnitude of the epidemic.

5.1.2. SIRS dynamics

Now let's be a bit more fancy and look at a mixture of the SIS and the SIR model called the SIRS model. This is a model for diseases in which infected attain immunity but then after sometime become susceptible again, so we have three reactions

$$S + I \xrightarrow{\alpha} 2I$$

$$I \xrightarrow{\beta} R$$

$$R \xrightarrow{\gamma} S$$

If we were to model this with regular ODEs we'd have

$$\begin{aligned} \dot{x} &= -\alpha xy + \gamma z \\ \dot{y} &= \alpha xy - \beta y \\ \dot{z} &= \beta y - \gamma z \end{aligned}$$

and with

$$1 = x + y + z$$

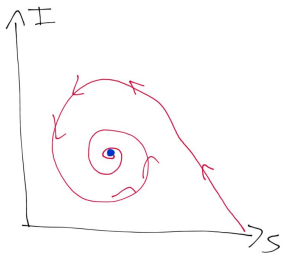
we are able to reduce to a 2d system

$$\begin{aligned} \dot{x} &= -\alpha xy + \gamma(1 - x - y) \\ \dot{y} &= \alpha xy - \beta y \end{aligned}$$

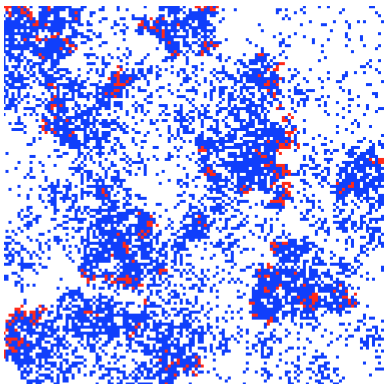
We can, as above, see that this system has a fixed point

$$x = \beta/\alpha \quad \text{and} \quad y = \frac{\gamma(1 - \beta/\alpha)}{\gamma + \beta}$$

which is only meaningful, if again $\beta < \alpha$. If we do a linear stability analysis of this fixed point, we will find that this is a stable spiral, so the system oscillates into its stable state.



But what does this look like when we look at it as a random lattice system? We see that also here we observe a critical transition if we change the ratio of β/α. However, we can see that the system can exhibit an interesting behavior as we get repetitive waves that go through the system. This is reminiscent of a phenomenon called avalanches which appear typically as a critical system approaches the critical point. We will come back to this phenomenon.



5.2. Spin systems

Epidemics aren't the only systems that exhibit critical or threshold dynamics. Let's look at a simple dynamical systems called a spin system which is borrowed from physics but it can be used to model social dynamics and consensus forming and opinion dynamics. Let's assume we have a lattice and each site has a variable

$$s_{ij}$$

which can either be -1 or 1 sometimes just denoted by a spin up or a spin down. Now let's look at the following simple dynamical system.

5.2.1. The random spin system

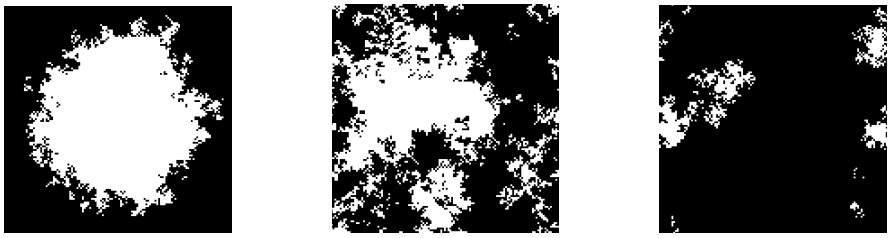
The simplest model is of course in which each spin randomly flips its opinion everytime it updates

$$s_i(t+1) = s_i(t)$$

If we have a system of N spins we can show that it will take prohibitively long for the system to reach consensus and even then it will go back to a non-consensus state. This is thus a stupid model. We need to incorporate interactions.

5.2.2. The Voter Model

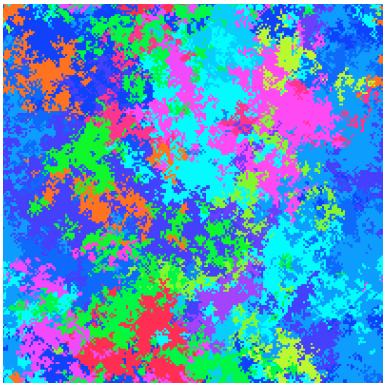
This model assume that at each step a spin looks at one of its neighbors, randomly chosen, and adapts its own opinion to agree with that neighbor. Let's see what this does, if we start with an intitial condition where we have a group of people having the same opinion in the middle surrounded by people with the opposing opinion. If we run this for some we we see that regions on all scales form that are in agreement and the border is fractal shaped. Interestingly, if we run this for a long time, eventually one opinion will prevail this is a consensus state.



This particular model has no parameters and in fact as we will see we can consider this to be a system at a crictical point which is why fluctuations are very large. We will see that a related model, the Ising model behaves just like that at the critical point.

5.2.3. The Multivoter Model

Let's generalize this to a model in which we have initially a set of M opinions on a lattice. We could describe this system by a spin that has M different directions and each spin aligns itself with the direction of one of the neighbors. Let's see what happens to them as a function of time. We observe that as time goes on, more and more opinions disappear from the system.

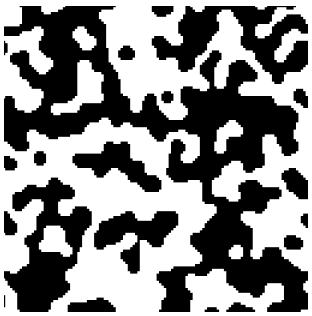


5.2.4. The peer rule

Let's see what patterns emerge if a local patch looks at what the majority of the neighborhood says. In other words, each spin computes

$$h_i = \sum_{j \in U_i} s_i$$

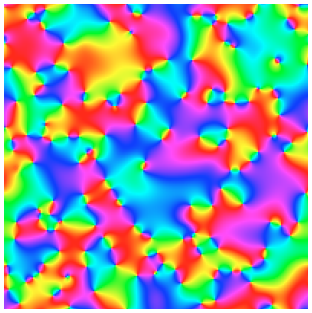
if $h_i > 0$ then $s_i(t+1) = 1$ and if $h_i < 0$ then $s_i(t+1) = -1$. Note that has no random component. It's a deterministic rule and we shouldn't be surprised that it will go into a fixed state of consensus region that have approximately the extent of the neighbor of the coupling. So unlike the voter model, the peer model will not go into a state of global agreement.



5.2.5. Peer rule with multiple opinions

Let's see if something similar happens if we have multiple opinions. In this case the sum above doesn't work so well. Instead, let's consider each opinion as an angle variable θ along the entire circle. And let's assume that each each spin when it updates an opinion, it does so according to the average opinion in its neighborhood:

We have seen this before. Again we see that this goes into a stable cinfiguration in which we see pinwheels, just like when we discussed oscillators and pattern-formation. And again, this is very different from the situation which we encounter if individual spins pick their opinion in alignment with a randomly chosen spin in its neighborhood.



5.3. The Ising model

One of the most important conceptual models in the context of social influence is the Ising Model. In this model, we assume that each node i in a network can be in one of two states

$$s_i = \begin{cases} 1 \\ -1 \end{cases}$$

These two states can represent two opposing political opinions. Now let's assume a network adjacency matrix quantifies the channels of interactions, $A_{ij} = 1$ means that nodes i and j can interact and according to the interaction change their state

$$s_i(t+1) \to -s_i(t)$$

based on their neighborhood. Let's assume that for each node, it is benefitial to adopt the opinion of its neighbors. Let's look at a pair s_i and s_j and the quantity

$$h_{ij} = -\frac{1}{2} s_i A_{ij} s_j.$$

Obviously, if these nodes are uncoupled this quantity is zero. If $A_{ij} = 1$ then

$$h_{ij} = \begin{cases} -1 & \text{if} \quad s_i = s_j \\ 1 & \text{if} \quad s_i = -s_j \end{cases}$$

Thus, if aligning s_i with s_j is benefitial, a negative h_{ij} is benefitial. Of course, node i is connected to many neighbors in general, and flipping it will align it with some and not others. But we can compute the net alignment by

$$h_i = -\frac{1}{2} s_i \sum_j A_{ij} s_j$$

We can rewrite this as

$$h_i = -\frac{1}{2}(k_+ - k_-)$$

where k_+ is the number of i's neighbors with $s_i = s_j$ and k_- with $s_j = -s_i$. Thus flipping i will induce a change in h_i of magnitude

$$\Delta h_i = -\frac{1}{2}(k_- - k_+) + \frac{1}{2}(k_+ - k_-) = -(k_- - k_+)$$

Thus, if

$$k_- > k_+$$

it is benefitial for i to switch because it lowers the overall agreement with i's neighborhood. The overall agreement in the entire network can be computed by

$$H = -\frac{1}{2}\sum_{ij} s_i A_{ij} s_j = -\frac{1}{2}\mathbf{s}^T \mathbf{A}\mathbf{s}$$

In physics this is the energy of a spin system that can model the behavior of magnetic spins. Clearly the lowest possible "energy" is when all spins are alighned in which case

$$H = -\frac{1}{2}\sum_{ij} A_{ij} = -L$$

where L is the number of links in the system.

We can now simulate this model on a computer by the following algorithm:

1. Pick a node at random
2. compute the change in "agreement" of i were to change it's state Δh_i
   a) if $\Delta h_i \leq 0$ flip the spin
   b) if $\Delta h_i > 0$ do not flip the spin
3. go back to 2.

This process will eventually go into the state that corresponds to full agreement in the set of nodes. How can we measure this? Simply by

$$M = \frac{1}{N}\sum_i s_i$$

so that $M = \pm 1$ if everyone agrees, and if the spins are random then $M \approx 0$. This model is not so interesting as such but becomes interesting in the following way:

Let's assume that each node has a probability to flip it's state, even if

$$\Delta h_i > 0$$

This is equivalent to going with the minority opinion. So we can set the system up

$$p(s_i \to -s_i) = \begin{cases} 1 & \text{if} \quad \Delta h_i \leq 0 \\ q_i & \text{if} \quad \Delta h_i > 0 \end{cases}$$

and we can chose

$$q_i = \exp(-\Delta h_i / T)$$

in which $T > 0$ is a parameter. If $T \to 0$ then $q_i = 0$ which is the case described above. If $T \gg \Delta h_i$ then $q_i \approx 1$ which means that I flip the spin with probability 1 as well. The question is, how does the overall opinion in the population look like as a function of the parameter T

$$M(T) = ?$$

This depends very much on the adjacency matrix A. In a regular two dimensional grid, the system exhibits a second order phase transition, for

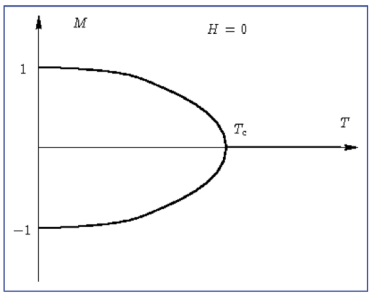
$$T > T_c$$

we find

$$M \approx 0$$

but for $T < T_c$

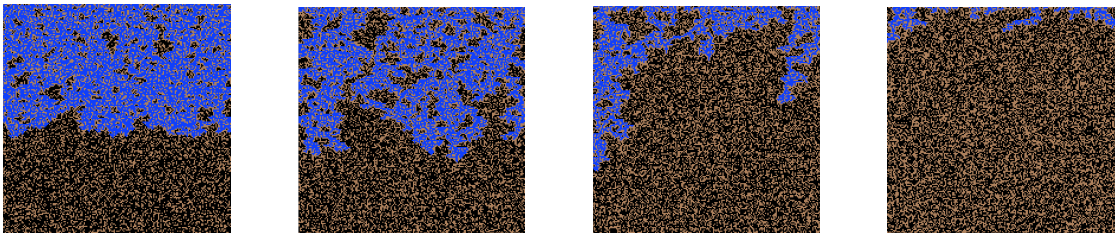
$$|M| > 0$$



The bottom line is that if the "noise" in the system is sufficiently large, then no consensus will form in a bi-modal opinion scenario, if random fluctations decrease then suddenly the system will go into one of the global consensus states.

5.4. Percolation on lattices

Another important class of systems in critical phenomena are percolation phenomena. The idea is the following. Given a medium that is disordered, in the simplest scenario consisting of "holes" and "grains", what is the critical concentration of holes p such that a liquid can percolate through the medium. This turns out to be a non-trivial question as we will see later. If we program this scenario in netlogo we see that the critical concentration for a liquid to percolate down to the bottom of the system is around $p_c = 0.64$. In the simulation below, at each step a "water" pixel will flow to a pixel below or sideways if that pixel is a hole. Obviously if $p = 0$ then water cannot go through, because everything is solid. if on the other hand $p = 1$ water will flow because it essentially falls through empty space.



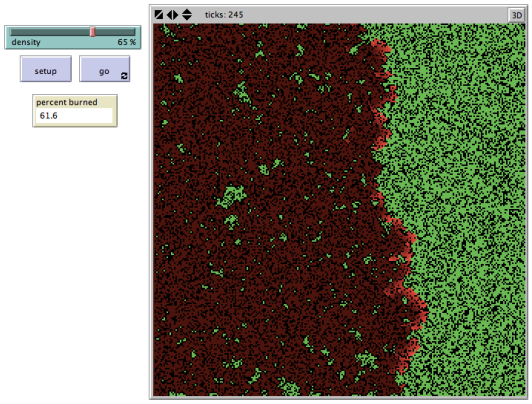
these frames are snapshots are various times for $p = 0.73, 0.65, 0.61$ ans 0.59. The last frame came to a halt after some time.

5.4.1. Forrest Fire

This is very much related to the idea of forrest fires which can be modelled using percolation theory. Here, we have a concentration of vegetation, for instance patches that contain trees and patches that do not. A forrest fire can move accross the entire plane if the concentration of trees is sufficiently large. Here als we find that the critical concentration of trees is aroung

$$p_c = 0.65$$

It is important to know that the behavior below and above the critical point is macroscopically different. If $p > p_c$ then the fire (or the percolating liquid) will pass through the entire system so the amount of vegetation in the forest fire model that is affected is of the order of the systemsize. If on the other hand $p < p_c$ only a tiny fraction (on average) is affected because the fire dies out. This is very much like the behavior we observed in the epidemic models where we also observed thresholds. Here the threshold is the concentration of patches that support the spreading phenomenon.



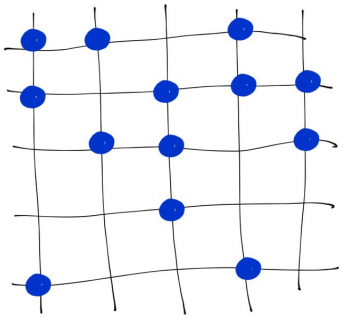
5.4.2. Site percolation

The systems above are so called site-percolation systems, where only a fraction p of possible sites exists. And in these systems we try to determing the critical fraction of

sites that must exist for percolation to occur. The actualy critical threshold for site percolation is given by

$$0.592746$$

which has been determined nuermically. To the isntructors knowledge the actual critical value is unknown on the square lattice.

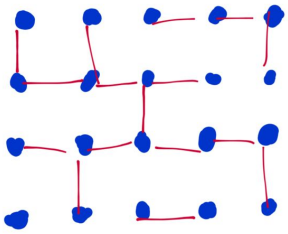


5.4.3. Bond percolation

There's another way to think about this. What if the physical process actually occurs along the bonds in a lattice, which may be the intuitive way to think of a liquid running through the system process. In this case we may ask, what is the fraction of links that are required to exist in the system such that it percolates. For the square lattice it's straighforward to compute this

$$p_c = 1/2$$

in other words as long as every lattice site has more than two links/bonds we are find.



5.4.4. Other Lattices

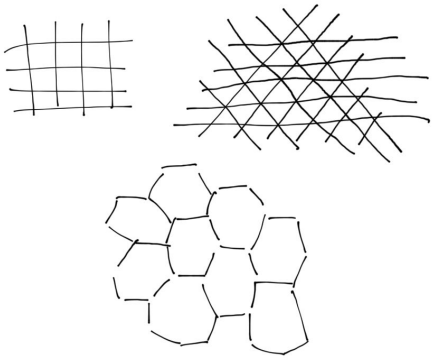
Sometimes, the physical situation requires to think of the sytem using other geometries, e.g. triangular lattices or honeycomb lattices. Sometimes in those other lattices it is

easier to compute the percolation threshold. For example the site percolation threshold in a triangular lattice is

$$p_c = 1/2$$

which means at least half of all possible sites have to exist for the system to percolate. However, the link percolation threshold is much more complicated to compute in a triangular lattice and is given by

$$p_c = 0.347296.$$



5.5. Networks

The idea of percolation is a very important concept in network science where regular lattices are replaced by random or in general more complex connectivity between sites. A network is just a collection of N nodes and L links. The adjacency matrix A tells us which two nodes are connected, so for nodes i and j we say $A_{ij} = 1$ if a link exists between i, j. The maximum number of links in such a network is given by

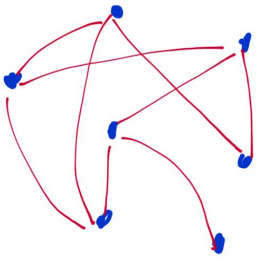
$$L_{\max} = \frac{N(N-1)}{2}$$

The maximum degree (number of connections to other nodes) of a node is $k_{\max} = N - 1$. In general the degree of a node is given by

$$k_i = \sum_j A_{ij}$$

and the mean degree across the network is

$$\langle k \rangle = \frac{1}{N} \sum_{i,j} A_{ij} = \frac{2L}{N}$$



5.5.1. ER Networks

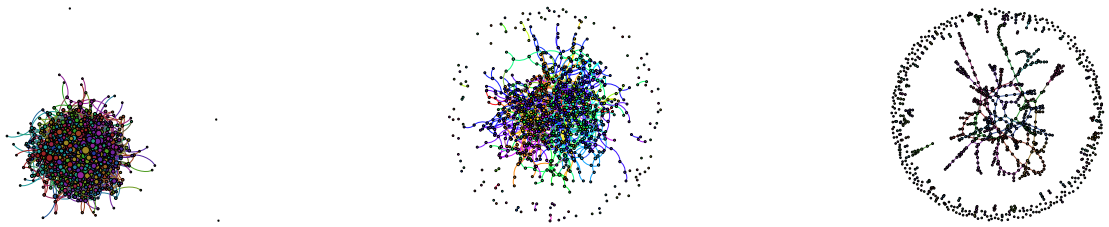
The simplest network is one in which we say that each one of the possible $N(N-1)/2$ links exists with a probability p. So the expected number of links in a network is

$$\langle L \rangle = pL_{\max} = p\frac{N(N-1)}{2}$$

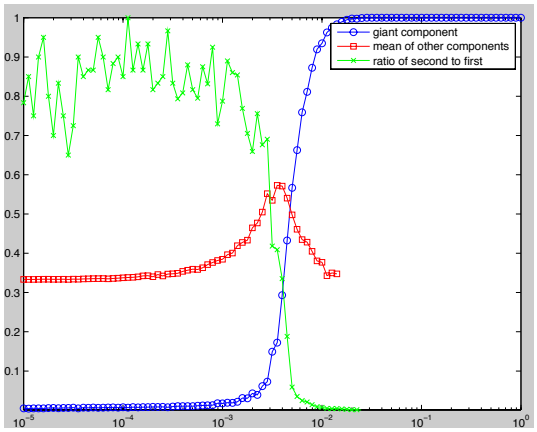
The expected degree is

$$\langle k \rangle = p(N-1).$$

The question is: What does the network look like as we decrease p from 1 to 0. We know that everything is connected for $p = 1$ and we know that everything is disconnected if $p = 0$. So it will make sense to observe as a function o p the number of components in such a network.



It turns out that here we have another critical phenomenon, which means that we can see a critical value $p_c$such that if $p > p_c$ the almost everything in the network is connected and if $p < p_c$ the network falls appart into many many small disconnected components. It's interesting to observe the transition though. The way this network disintegrates is that little pieces get disconnected from a Giant component that contains almost all the nodes. It's not like the system falls appart continuously.



One can use a little network theory to show that a network falls appart if

$$\frac{\langle k^2 \rangle}{\langle k \rangle} =< 1$$

We can derive the degree distribution in an ER network and it is given by a binomial distribution

$$P(k) = \begin{pmatrix} N-1 \\ k \end{pmatrix} p^k (1-p)^{N-1-k}$$

so the above inequality becomes

$$1 + (N-2)p < 1$$

so

$$p < \frac{2}{N-2}$$

which we can write as

$$\frac{\langle k \rangle}{N-1} < \frac{2}{N-2}$$

so if

$$\langle k \rangle < 2$$

which makes sense, every node has to be connected to at least to other nodes for the giant cluster to connect everywhere. This also means that we have

$$L \approx N.$$

5.5.2. Spreading processes through networks

This is some very important information if we want to mitigate the spread of a disease throughout a network. Let's assume that we have a network with a degree

$$\langle k \rangle > 2$$

Then a key question then is what fraction Q of links need to be removed in order go below the critical value $k_c = 2($

$$(1 - Q) \langle k \rangle = 2$$

$$1 - Q = 2/\langle k \rangle$$

$$Q = 1 - \frac{2}{\langle k \rangle}$$

So for instance if $\langle k \rangle = 5$ then

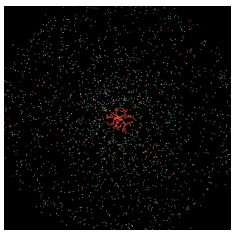
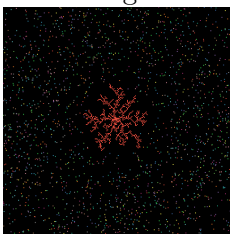
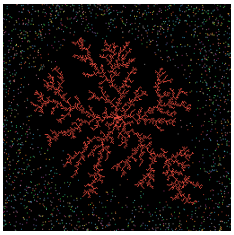
$$Q = 60\%$$

6. Growth and Aggregation Processes

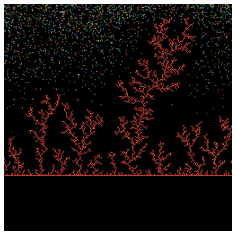
A number of processes in nature are transient in the sense that a pattern or a structure changes by growing without losing some of its characteristic statistical features. In fact the appearance of some structure are driven by an underlying growth process. In the intriduction we discussed that e.g. christals grow, trees grow and a bunch of other things. There are many processes the rely on the mechanism of aggregation which means that a structure grows by outside things permanently attaching to it. Christals and snowflakes are a good example of this or systems driven by precipitation. Other structures grow from the inside by sprouting substructures. Trees and many biological systems are of this type.

6.1. Aggregation processes

The most basic but also already quite interesting aggregation process is called diffusion limited aggregation. It works as follows. We start with a seed of a structure of small volume or area. Then we assume that this seed is exposed to a "bath" of particles that perform random walks. Say in two dimensions. The clue is now, that we assume that whenever such a particle gets sufficiently close to the seed structure that it attaches and add a little area or volume to the structure on the surface. If we have an infinite supply of particles and if the density is small enough that they don't interact (except for with the growing structure, the key question is what is that structure going to look like. IF we simuate this with netlogo we see something interesting emerge

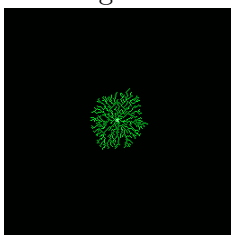
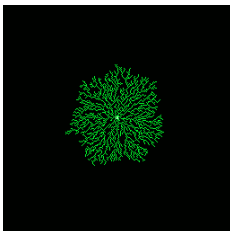
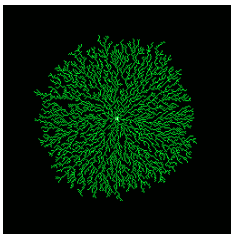




We could suscpect that this pattern would depend much on say the size and shape of the diffusing particles and for example the diffusion coefficient but it turns out that the fractal like shape is very generic and robust against chances in parameters. Sometimes the geometry can dictate the overall geometry of the structure. Here's an example if we start injecting particles into a system and the initial structure is a line:



6.1.1. attractive aggregation

The key mechanism behind these structures is that the attachment process introduces a memory in the system in the sense that after an attachment potential paths are blocked which is why these gaps in between the branches emerge. The randomness of the moving particles is actually not necessary for this as the following example illustrates. If we seed the system again with a smal structure in the center of the system around the origin and randomly start particles at a radius R and let them move along straight lines to the origin we may expect that a disk will emerge. But the diffusing particles have a finite size and if their initial placement is random along the circle of radius R the structure that emerges is more similar to the above than to a filled disk:

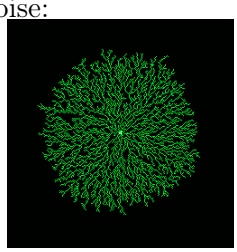
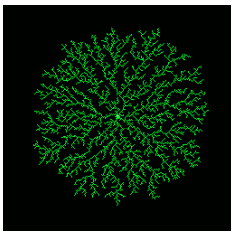




6.1.1.1. Attractive aggregation with noise

We can carefully compare the relative impact of noise and attractive motion of the particles by modelling the motion of the particles by a langevin equation

$$d\mathbf{X} = -\gamma \mathbf{X} dt + \sigma d\mathbf{W}$$

where the first terms makes the particles relax to the origin and the σ is the noise in the system. Here are two images, on the left one without noise and on the right with finite noise:

If we inspect these figures carefully then we see that the one on the right is less dense. Which is interesting because one could expect that diffusing particles have a higher

likelihood of going into the non occupied spaces. But this is not the case. Of course we need a way to quantify this, we will do this in a minute.

6.1.2. General dynamics and aggregation

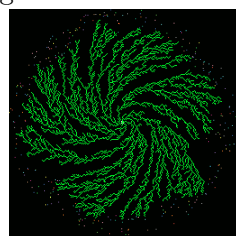
We can generalize the system above of course to model aggregation of particles that obey general dynamical equations

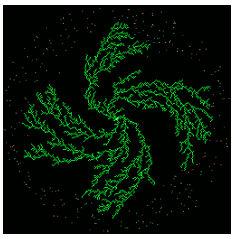
$$d\mathbf{X} = -f(\mathbf{X})dt + \sigma d\mathbf{W}$$

where we have a determinstic part and noise. Let's look at this example

$$\begin{aligned} dx &= -\gamma x dt - \delta y dt + \sigma dW_1 \\ dx &= -\gamma y dt + \delta x dt + \sigma dW_2 \end{aligned}$$

without the noise and $\gamma, \delta > 0$ the particles will spiral towards the origin. Depending on the ration of δ/γ we see that the spiraling motion of the aggregating particles leaves a footprint on the emergent structure. The image on the left has no noise, the one on the right finite noise





And again we see that if noise is present then the structure is sparser. How can we measure this?

6.1.3. Fractal Dimension

The concept of fractal dimension helps us here. There are different ways of defining a fractal dimension. We will be using the concept introduced by Haussdorff. Let's start with the idea or ordinary dimension. Let's think about two a closed region embedded in a space. For instance some sort of membrane in three dimensions.

We know that the boundary of the membrane is a line and has dimension 1. The surface of the membrane has dimension 2. But how can we figure this out? Let's do a common sense approach. Let start with a bunch of beads or radius Δx. Now let's cover the contour with beads such that they align along and cover the entire contour. Now let's count them and say we've got$N(\Delta x)$ beads. Now if we decrease the radius of the beads we or course will have more beads to align along the contour. In face if we half the radius Δx we have to double the number of beads, so

$$N(\Delta x) \sim \Delta x^{-1}.$$

Now let's assume that we want to cover the membrane with beads and for a given Δx we find we need $N_M(\Delta x)$ beads. If we decrese the radius say by a factor of 2 we need of course 4 times as many beads, in fact

$$N_m(\Delta x) \sim \Delta x^{-2}$$

In fact for a D dimensional object embedded in some higher-dimensional space we need

$$N(\Delta x) \sim \Delta x^{-D}$$

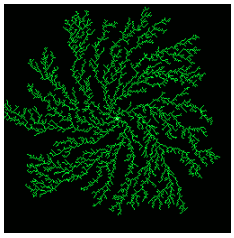
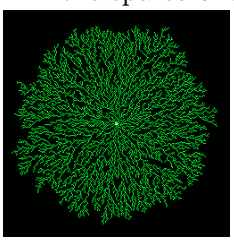
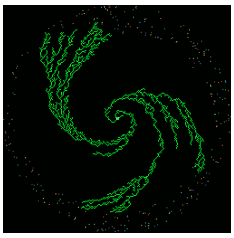
beads. The idea is now to define the dimension of an object as

$$D = -\lim_{\Delta x \to 0} \frac{\log N(\Delta x)}{\Delta x}$$

For smooth lines or surfaces or volumes this is straighforward. But what happens in the above DLA examples? We can do this experiment in netlogo actually and measure D by chanding Δx and measure $N(\Delta x)$. We know that the DLA structures have to have a dimension smaller or equal to 2 becuase they exist in the plane. But when we do the measurement we find that

$$1 < D < 2$$

so these objects have a fractal dimension. This example on the left e.g. has $D = 1.3626$ and the one in the middle has $D = 1.41$ the sparse one on the right has $D = 1.208$.

6.2. Cellular Automata

Let's now look at a particular subclass of systems where the opposite dynamics occurs. We essentially start with a seed but this seed can spawn new children and also, depending on it's interaction die. Let's first thing of a one dimensional system in which we have lattice site that can be occupied or not occupied. So the state of a lattice site is either 0 or 1.

6.2.1. Example: Competion Coorperation

Let's assume that

1. a site that is 0 remains zero if surrounded by two zeros. this means that nothing come from nothing.

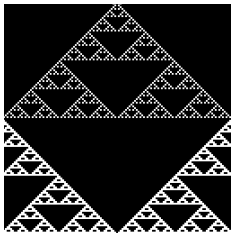
2. It a site is 0 and but we've got one neighbor that is 1 the next state is 1, so we interpret this as the 1 invading into the vacant site
3. If the site is 0 and both neighbors are 1 they compete for the vacant site and will not occupy it..
4. If the site is occupied and has only one occupied neighbor the site will remain occupied, e.g. by cooperation
5. If on the other hand the site is occupied and has two empty neighbors the occupied site dies.
6. If the site is occupied and has occupied neighbors the site dies, too.

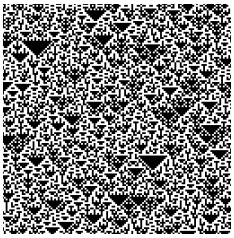
We can write is in table form, where the first row is the state around the middle referemce site and the bottom row it's state in the next iteration.

1**1**1	1**1**0	1**0**1	1**0**0	0**1**1	0**1**0	0**0**1	0**0**0
0	1	0	1	1	0	1	0

We can easily run this in netlogo. The pattern that emerges depends very much on the intitial condition. Below are the results for a system that starts out with a single occupied site in the top pixel row and time evolves towars the bottom. The right figure is the same with an initial condition that is randomly assigning occupation to pixels in the top row.

We have seen the structure before, it's the Sierpiensky gasket.

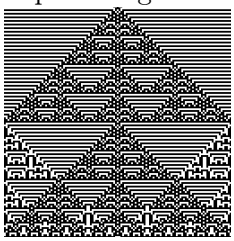


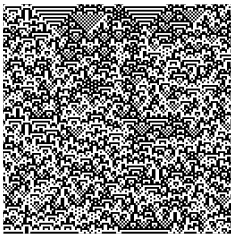


Let's now modify the dyanmics a little bit and assume the following rule table

1**1**1	1**1**0	1**0**1	1**0**0	0**1**1	0**1**0	0**0**1	0**0**0
0	1	1	0	1	0	0	1

How do we interpret this? Well, this means that on empty sites with no occupied neighbors in site new occupation will occur. This can also happen if one of the neighbors is occupied. but not if only one is occupied and not if the environment is too overcrowded. The pattern generated by this dynamics looks a bit different.

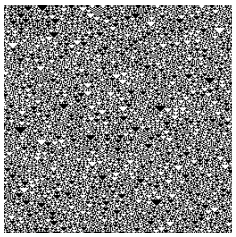




Let's try another system

1**1**1	1**1**0	1**0**1	1**0**0	0**1**1	0**1**0	0**0**1	0**0**0
1	0	0	1	0	1	1	0

and the pattern looks like this:



6.2.2. Elementary cellular automata

The table notation above introduces a systematic way of defining particular rule-sets. Note that each patch and its 2 neighbors can have $2^3 = 8$ different configurations. And the rules are defined by mapping these 8 configurations to the next state of the center patch. If we arrange the inputs as in the table above, the result is a binary string, e.g. in the last example

$$s = 10010110$$

How many different such strings exist, i.e. how many rules can we have at most? That is

$$2^{(2^3)} = 2^8 = 256$$

and we can count them by transforming the above binary sequence into a decimal number the above is

$$\begin{aligned} rule &= 1 \times 2^7 + 0 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0 \\ &= 128 + 16 + 4 + 2 = 150 \end{aligned}$$

The other above are rules are

$$R = 90$$

and

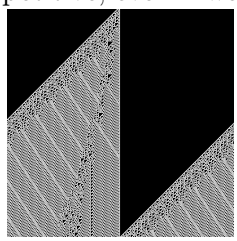
$$R = 105$$

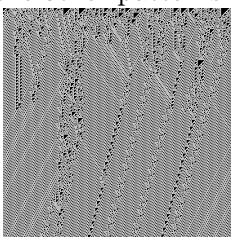
6.2.2.1. Example $R = 110$

Now let's pick a rule number and write down the table for it

1**1**1	1**1**0	1**0**1	1**0**0	0**1**1	0**1**0	0**0**1	0**0**0
0	1	1	0	1	1	1	0

This is rule 110, a very famous rule. Let's look at the pattern. This pattern is non repetitive, even if we start with a single occupied site. Different from the other patterns:





6.2.3. Stochastic Cellular automata

It's of course straightforward to generalize the above to a system in which we specify the dynamics by a table

1**1**1	1**1**0	1**0**1	1**0**0	0**1**1	0**1**0	0**0**1	0**0**0
p_7	p_6	p_5	p_4	p_3	p_2	p_1	p_0

in which the the states are no longer deterministic but probabilities that given the three patch state, the next state will be occupied. For example if we look at the system

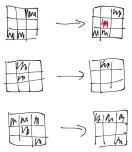
1**1**1	1**1**0	1**0**1	1**0**0	0**1**1	0**1**0	0**0**1	0**0**0
0	0.25	0	0.5	0.25	1	0.5	0

we see a structure like this:

6.2.4. The game of life.

The most famous cellular automaton is the game of life. This automaton evolves in two dimensions and is to resemble or model some of the basic mechanism on life: overcrowding, reproduction, competition etc. The rules are very simple. A lattice site can be occupied $s = 1$ or empty, $s = 0$. Every patch updates its state in parallel and synchronous with all other patches aacording to the following basic rules that can change the state

1. $s = 0 \quad \rightarrow \quad s = 1$: only if exactly three neighbors have $s = 1$
2. $s = 1 \quad \rightarrow \quad s = 0$: if there are less than 2 neighbors alive (under-population) and if more than 3 are alive (over population)



This means that for instance if for instance a live cell has 2 or three neighbors it remains alive. An empty cell with anything but 3 neighbors remains empty. This simple automaton has an amazingly rich intrinsic potential for pattern generation as we can explore using netlogo. One of the key feature of this simple rule set is that it can

1. Produce complex dynamic patterns
2. Produce oscillatory behavior
3. Evolve subsystems that reproduce copies
4. be designed to mimic life

The best way to explore the power of the game of life is using the netlogo program that comes along with these notes.

6.3. Lindenmeyer system (L-systems)

One of the most famous approaches to modelling natural patterns, in particular patterns that grow are Lindenmeyer systems that we have briefly discussed in the introduction. Formally Lindenmeyer systems are an abstract set of

1. Variables
2. Constants
3. Rules
4. plus one initial condition

At each step, in a Lindenmeyer system, the rules inform how to replace variables with new states. The best way to understand this is by looking at an example:

6.3.1. A simple L-System

Let's say we have variables X and Y and a constant denoted by the symbol α. Let's denote the state of the L-system by S_n after n iterations and an initial condition

$$S_0 = X$$

Let's now specify the rules

$$\begin{aligned} X &\rightarrow Ya \\ Y &\rightarrow aX \end{aligned}$$

Using this we can compute

$$\begin{aligned} S_1 &= Ya \\ S_2 &= aXa \\ S_3 &= aYaa \end{aligned}$$

etc. That's all. The interesting aspect of a Lindenmeyer system comes, if we attach a meaning to the constants and the variables. Let's look at another example:

6.3.2. The Sierpinsky gasket, again

Let's say we have two variables X and Y and constant α and β, the initial condition is

$$S_0 = X$$

and the rules are

$$\begin{aligned} X &\rightarrow YaXaY \\ Y &\rightarrow XbYbX \end{aligned}$$

Let's write down a few iterations

$$\begin{aligned} S_1 &= YaXaY \\ S_2 &= XbYbXaYaXaYbXbYbX \\ S_3 &= .... \end{aligned}$$

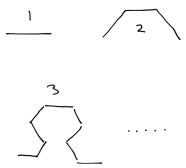
Now let's see what structure this can describe if we associate geometric structures to it. Let's assume that

$$X, Y: \quad \text{draw a line of length} \quad 1/2^n$$

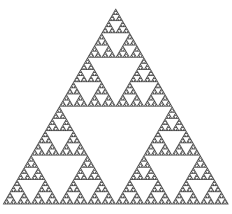
and

$$a, b: \quad \text{turn left/right 60 deg. respectively}$$

Then we get the following iterations:



We see what emerges is the Sierpinsky triangle. We also saw this as a result of rule 90 in the cellular automaton.



6.3.2.1. Fractal dimension

Looking at this self-similar structure as a L-system allows us to directy compute the fractal dimension of this structure. Drawing more and more shorter and shorter lines means that we need to look at how the number of lines increases with n and comparing to the length of the individual segments. So the number or lines is

$$N(n) = 3^n$$

Their length decreases with

$$\Delta x = \left(\frac{1}{2}\right)^n = 2^{-n}$$

So the "normal" length

$$N\Delta x = \left(\frac{3}{2}\right)^n$$

is divergent. How we do something similar to what we did in the DLA fractals We would like to derive how the N scales with Δx so

$$N \sim \Delta x^{-D}$$

We have

$$\frac{-\log \Delta x}{\log 2} = n$$

So

$$N(\Delta x) = 3^{-\frac{\log \Delta x}{\log 2}} = \Delta x^{-D}$$

where

$$D = \log 3 / \log 2 = 1.5849 > 1$$

so the Haussdorf dimension of the Sierpiensky triangle is this.

6.3.3. The Dragon curve

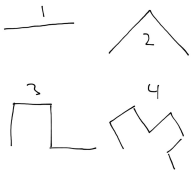
Here's another one quite similar. Again we've got variables X and Y and constants a, b, the initial condition $S_0 = X$ but this time the rules

$$\begin{aligned} X &\rightarrow XaY \\ Y &\rightarrow XbY \end{aligned}$$

which generates the sequence

$$\begin{aligned} S_1 &= XaY \\ S_2 &= XaYaXbY \\ S_3 &= XaYaXbYaXaYbXbY \end{aligned}$$

Now we interpret X, Y as "draw line" of length $1/\sqrt{2}^n$ and a, b turn at a right angle left and right respectively. This is what it looks like



and after many iterations we get what is known as the dragon curve. What's the dimension of this guy? At each step we double the number of lines so

$$N(n) = 2^n$$

The lines decrease by

$$\Delta x \sim 2^{-n/2}$$

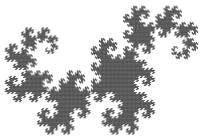
so

$$n = -2 \log \Delta x / \log 2$$

so

$$N(\Delta x) = 2^{-2 \log \Delta x / \log 2} = \Delta x^{-2}$$

Interesting, the fractal dimension of this curve is 2 which means this guy is space filling.



6.3.4. The Koch Snowflake

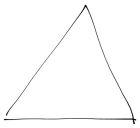
Let's look at this one. This time only one variable and one constant

$$X \to XaXaaaXaX$$

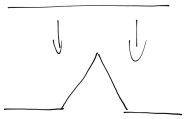
and an intial state

$$S_0 = XaaXaaX$$

Now we interpret X as draw a line of length 3^{-n} and a turn left 60 degrees. So S_0 is a triangle.



The rule means that each line is preplaced by a structure like this



Again, we would like to know the fractal dimension. we have

$$N \sim 4^n$$

and

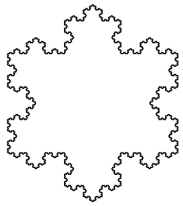
$$\Delta x \sim \left(\frac{1}{3}\right)^n$$

so

$$N(\Delta x) \sim 4^{-\log \Delta x / \log 3} = \Delta x^{-\log 4 / \log 3}$$

so

$$D = \frac{\log 4}{\log 3} = 1.26185$$



6.3.5. Trees and Branching

An important aspect of self-similar structures are branching events, in particular in plants this is observed. In L-systems this is easily done. Let's look at this system, Let's assume we have two variables L and B where L means leaf and B means branch. As a tree grows each leaf is replaced by a branch from which two leafs grow at 2 different angles α and β relative to the original leaf. An stems grow. We start with one leaf, so the initial condition is

$$S_0 = L$$

One rule is to make branches double in size so

$$B \to BB$$

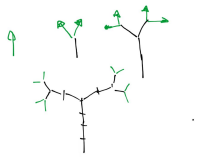
Now the more complicated rule

$$L \to B((\alpha L)\beta L)$$

We have used here another symbol, the paranthesis. This essentially means that (memorizes the current position and) goes back to it. So this particular rule has iterations

$$\begin{aligned} S_1 &= B((\alpha L)\beta L) \\ &= BB((\alpha B((\alpha L)\beta L)\beta B((\alpha L)\beta L)) \end{aligned}$$

Now this can become very complex, but a computer can read this very well.



6.3.5.1. A more complex tree

Here's another example that follows similar lines of thinking. We start with a leaf

$$S_0 = L$$

and we have one rule

$$L \to \delta B(((\alpha L)\beta L)\gamma L)$$

This rule replaces each leaf with a branch and three leafs. The additional constant δ could be interpreted has replacing each leaf with a branch that is slightly shorted and the other parameters α, β, γ could encode for an angle and the length of the sprouting leaf. With this one can generate different types of morphologies.

7. Collective motion.

7.1. The Viszek Model

Recall the following system that we discussed in critical phenomena. We assumed that we had a lattice of spins. We discussed one system in which each spin had an angle variable

$$\theta_i(t)$$

And we said we can update the spins according to

$$\theta_i(t + \Delta t) = \langle \theta_i(t) \rangle$$

so each theta set to the average orientation of its neighbors. This generated the typical pinwheel patterns:

We can modify this slightly and at each iteration step we add a little bit of noise

$$\theta_i(t + \Delta t) = \langle \theta_i(t) \rangle + \xi$$

where ξ is a random number chose in some range $[-\eta, \eta]$. What we observe is the following a noise version of the above system which is not suprising.

This is the basis for the simplest model for collective motion. Imagine that the patches are now turtles that can move. So we have a set of turtles labeled i and each turtle has a position $\mathbf{x}_i(t)$ and a velocity $\mathbf{v}_i(t)$. Imagine that all turtles are moving freely so that

$$\mathbf{x}_i(t + \Delta t) = \mathbf{x}_i(t) + \mathbf{v}_i(t)\Delta t$$

This means that all turtles are moving in the direction

$$\mathbf{n}_i = \frac{\mathbf{v}_i(t)}{v_i}$$

where v_i is turtle i's speed. The velocity can be decomposed into the speed and the heading

$$\mathbf{v}_i(t) = v_i(t) \begin{pmatrix} \cos\theta_i(t) \\ \sin\theta_i(t) \end{pmatrix}$$

We now assume that the speed of the turtles does not change

$$v_i(t) = v_0 = \text{const.}$$

The only way that the velocity changes is by changing the headings. We assume that

$$\theta_i(t + \Delta t) = \langle \theta(t) \rangle_R$$

where

$$\langle \theta(t) \rangle_R$$

is the average heading of turtles in the neighborhood of turtle i. If we start in a finite area with periodic boundary conditions we see that this system evolves into a state where every turtle will eventually move into one direction. This is not so interesting. what if we modify the system to

$$\mathbf{x}_i(t + \Delta t) = \mathbf{x}_i(t) + \mathbf{v}_i(t)\Delta t$$
$$\theta_i(t + \Delta t) = \langle \theta(t) \rangle_R + \xi$$

where $\xi \in [-\eta, \eta]$. This system has three parameters, the overall velocity v_0 the noise η and the density of turtles $\rho = N/L^2$. We can now monitor the behavior of the collective as a function of these parameters. The way we quantify this behavior is by looking at the quantity

$$\phi = \frac{1}{N v_0} \left| \sum_i \mathbf{v}_i(t) \right|$$

if all the velocities are alligned we have $\phi = 1$ if they scatter then $\phi \approx 0$. For fixed ρ and fixed v_0 we see that there's sharpt transition between ordered swarm like behavior and the disordered state.. The same happens for fixed noise and increasing the density.

In fact this model has a second order phase transition which we measure by the order parameter ϕ..

7.1.1. Variable speed viszek model

The above model has been varied in a number of ways to model specific mechanisms that may play a role in flocking behavior. In one varient the speed also changes as a function of the consensus of the neighborhood of a turtle. Let's say we have a local order parameter

$$\phi_i = \frac{1}{M_i v_0} \left| \sum_{j \in U(i)} \mathbf{v}_j(t) \right|$$

and each turtle measures how aligned its neighbors are and chooses a speed accordinly, so

$$v_i(t + \Delta t) = v_0 \phi_i^\gamma$$

where γ is a new parameter. This means, the more aligned the neighbors, the higher the speed. What does this look like? This introduces a shearing in the stream of turtles and a greater flexibility of the entire swarm.

7.1.2. The ego and the group

Quite often an additional parameter is introduced to quantify the impact of the direction of the reference turtle and the average of the environment. In this case the update rul for the angle is

$$\theta_i(t + \Delta t) = \alpha \theta_i(t) + (1 - \alpha) \langle \theta(t) \rangle_R + \xi$$