10

ONE-DIMENSIONAL MAPS

10.0 Introduction

This chapter deals with a new class of dynamical systems in which time is *discrete*, rather than continuous. These systems are known variously as difference equations, recursion relations, iterated maps, or simply ***maps***.

For instance, suppose you repeatedly press the cosine button on your calculator, starting from some number x_0. Then the successive readouts are $x_1 = \cos x_0$, $x_2 = \cos x_1$, and so on. Set your calculator to radian mode and try it. Can you explain the surprising result that emerges after many iterations?

The rule $x_{n+1} = \cos x_n$ is an example of a ***one-dimensional map***, so-called because the points x_n belong to the one-dimensional space of real numbers. The sequence $x_0, x_1, x_2, \ldots$ is called the ***orbit*** starting from x_0.

Maps arise in various ways:

1. *As tools for analyzing differential equations.* We have already encountered maps in this role. For instance, Poincaré maps allowed us to prove the existence of a periodic solution for the driven pendulum and Josephson junction (Section 8.5), and to analyze the stability of periodic solutions in general (Section 8.7). The Lorenz map (Section 9.4) provided strong evidence that the Lorenz attractor is truly strange, and is not just a long-period limit cycle.
2. *As models of natural phenomena.* In some scientific contexts it is natural to regard time as discrete. This is the case in digital electronics, in parts of economics and finance theory, in impulsively driven mechanical systems, and in the study of certain animal populations where successive generations do not overlap.
3. *As simple examples of chaos.* Maps are interesting to study in their own right, as mathematical laboratories for chaos. Indeed, maps are capable

of much wilder behavior than differential equations because the points x_n *hop* along their orbits rather than flow continuously (Figure 10.0.1).

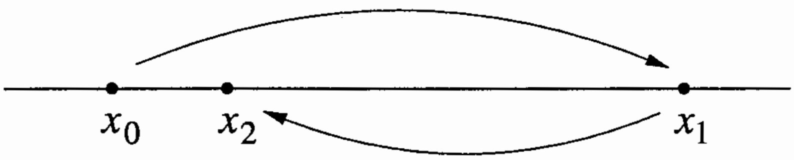

Figure 10.0.1

The study of maps is still in its infancy, but exciting progress has been made in the last twenty years, thanks to the growing availability of calculators, then computers, and now computer graphics. Maps are easy and fast to simulate on digital computers where time is *inherently* discrete. Such computer experiments have revealed a number of unexpected and beautiful patterns, which in turn have stimulated new theoretical developments. Most surprisingly, maps have generated a number of successful predictions about the routes to chaos in semiconductors, convecting fluids, heart cells, lasers, and chemical oscillators.

We discuss some of the properties of maps and the techniques for analyzing them in Sections 10.1–10.5. The emphasis is on period-doubling and chaos in the logistic map. Section 10.6 introduces the amazing idea of universality, and summarizes experimental tests of the theory. Section 10.7 is an attempt to convey the basic ideas of Feigenbaum's renormalization technique.

As usual, our approach will be intuitive. For rigorous treatments of one-dimensional maps, see Devaney (1989) and Collet and Eckmann (1980).

10.1 Fixed Points and Cobwebs

In this section we develop some tools for analyzing one-dimensional maps of the form $x_{n+1} = f(x_n)$, where f is a smooth function from the real line to itself.

A Pedantic Point

When we say "map," do we mean the function f or the difference equation $x_{n+1} = f(x_n)$? Following common usage, we'll call *both* of them maps. If you're disturbed by this, you must be a pure mathematician . . . or should consider becoming one!

Fixed Points and Linear Stability

Suppose x^* satisfies $f(x^*) = x^*$. Then x^* is a ***fixed point***, for if $x_n = x^*$ then $x_{n+1} = f(x_n) = f(x^*) = x^*$; hence the orbit remains at x^* for all future iterations.

To determine the stability of x^*, we consider a nearby orbit $x_n = x^* + \eta_n$ and ask whether the orbit is attracted to or repelled from x^*. That is, does the deviation η_n grow or decay as n increases? Substitution yields

$$x^* + \eta_{n+1} = x_{n+1} = f(x^* + \eta_n) = f(x^*) + f'(x^*)\eta_n + O(\eta_n^2).$$

But since $f(x^*) = x^*$, this equation reduces to

$$\eta_{n+1} = f'(x^*)\eta_n + O(\eta_n^2).$$

Suppose we can safely neglect the $O(\eta_n^2)$ terms. Then we obtain the *linearized map* $\eta_{n+1} = f'(x^*)\eta_n$ with *eigenvalue* or ***multiplier*** $\lambda = f'(x^*)$. The solution of this linear map can be found explicitly by writing a few terms: $\eta_1 = \lambda\eta_0$, $\eta_2 = \lambda\eta_1 = \lambda^2\eta_0$, and so in general $\eta_n = \lambda^n\eta_0$. If $|\lambda| = |f'(x^*)| < 1$, then $\eta_n \to 0$ as $n \to \infty$ and the fixed point x^* is ***linearly stable***. Conversely, if $|f'(x^*)| > 1$ the fixed point is ***unstable***. Although these conclusions about local stability are based on linearization, they can be proven to hold for the original nonlinear map. But the linearization tells us nothing about the ***marginal*** case $|f'(x^*)| = 1$; then the neglected $O(\eta_n^2)$ terms determine the local stability. (All of these results have parallels for differential equations—recall Section 2.4.)

EXAMPLE 10.1.1:

Find the fixed points for the map $x_{n+1} = x_n^2$ and determine their stability.

Solution: The fixed points satisfy $x^* = (x^*)^2$. Hence $x^* = 0$ or $x^* = 1$. The multiplier is $\lambda = f'(x^*) = 2x^*$. The fixed point $x^* = 0$ is stable since $|\lambda| = 0 < 1$, and $x^* = 1$ is unstable since $|\lambda| = 2 > 1$. ■

Try Example 10.1.1 on a hand calculator by pressing the x^2 button over and over. You'll see that for sufficiently small x_0, the convergence to $x^* = 0$ is *extremely* rapid. Fixed points with multiplier $\lambda = 0$ are called ***superstable*** because perturbations decay like $\eta_n \sim \eta_0^{(2^n)}$, which is much faster than the usual $\eta_n \sim \lambda^n\eta_0$ at an ordinary stable point.

Cobwebs

In Section 8.7 we introduced the ***cobweb*** construction for iterating a map (Figure 10.1.1).

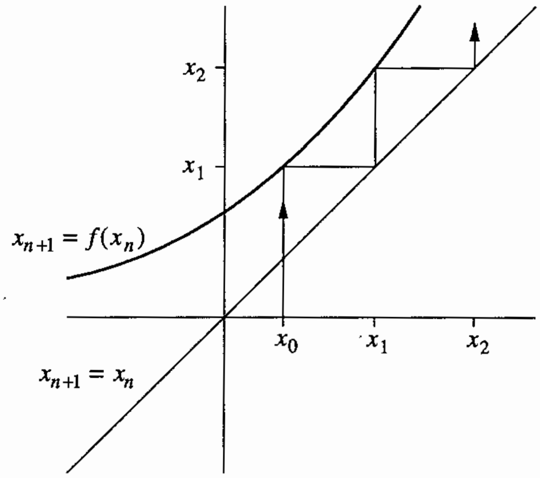

Figure 10.1.1

Given $x_{n+1} = f(x_n)$ and an initial condition x_0, draw a vertical line until it intersects the graph of f; that height is the output x_1. At this stage we could return to the horizontal axis and repeat the procedure to get x_2 from x_1, but it is more convenient simply to trace a horizontal line till it intersects the diagonal line $x_{n+1} = x_n$, and then move vertically to the curve again. Repeat the process n times to generate the first n points in the orbit.

Cobwebs are useful because they allow us to see global behavior at a glance, thereby supplementing the local information available from the linearization. Cobwebs become even more valuable when linear analysis fails, as in the next example.

EXAMPLE 10.1.2:

Consider the map $x_{n+1} = \sin x_n$. Show that the stability of the fixed point $x^* = 0$ is not determined by the linearization. Then use a cobweb to show that $x^* = 0$ is stable—in fact, *globally* stable.

Solution: The multiplier at $x^* = 0$ is $f'(0) = \cos(0) = 1$, which is a marginal case where linear analysis is inconclusive. However, the cobweb of Figure 10.1.2 shows that $x^* = 0$ is locally stable; the orbit slowly rattles down the narrow channel, and heads monotonically for the fixed point. (A similar picture is obtained for $x_0 < 0$.)

To see that the stability is global, we have to show that *all* orbits satisfy $x_n \to 0$. But for any x_0, the first iterate is sent immediately to the interval $-1 \le x_1 \le 1$ since $|\sin x| \le 1$. The cobweb in that interval looks qualitatively like Figure 10.1.2, so convergence is assured. ■

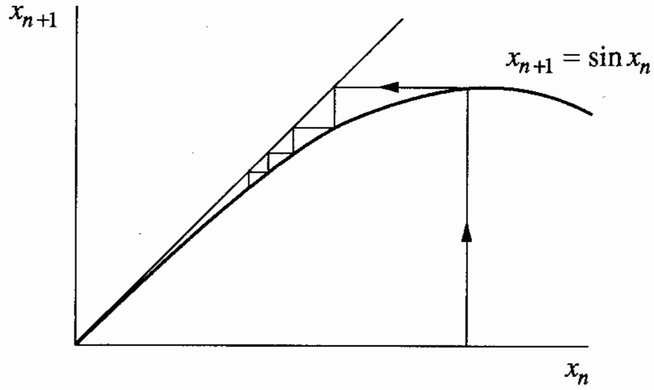

Figure 10.1.2

Finally, let's answer the riddle posed in Section 10.0.

EXAMPLE 10.1.3:

Given $x_{n+1} = \cos x_n$, how does x_n behave as $n \to \infty$?

Solution: If you tried this on your calculator, you found that $x_n \to 0.739\ldots$, no matter where you started. What is this bizarre number? It's the unique solution of the transcendental equation $x = \cos x$, and it corresponds to a fixed point of the map. Figure 10.1.3 shows that a typical orbit spirals into the fixed point $x^* = 0.739\ldots$ as $n \to \infty$. ■

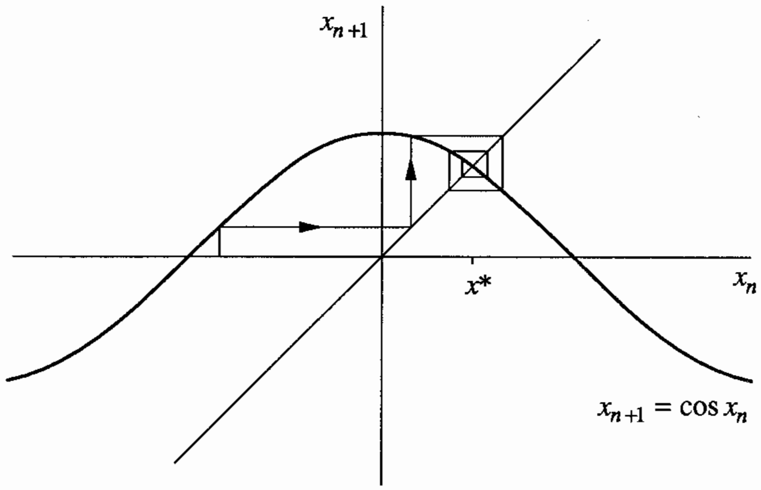

Figure 10.1.3

The spiraling motion implies that x_n converges to x^* through *damped oscillations*. That is characteristic of fixed points with $\lambda < 0$. In contrast, at stable fixed points with $\lambda > 0$ the convergence is monotonic.

?

10.2 Logistic Map: Numerics

In a fascinating and influential review article, Robert May (1976) emphasized that even simple nonlinear maps could have very complicated dynamics. The article ends memorably with "an evangelical plea for the introduction of these difference equations into elementary mathematics courses, so that students' intuition may be enriched by seeing the wild things that simple nonlinear equations can do."

May illustrated his point with the ***logistic map***

$$x_{n+1} = rx_n(1 - x_n), \qquad (1)$$

a discrete-time analog of the logistic equation for population growth (Section 2.3). Here $x_n \geq 0$ is a dimensionless measure of the population in the nth generation and $r \geq 0$ is the intrinsic growth rate. As shown in Figure 10.2.1, the graph of (1) is a parabola with a maximum value of $r/4$ at $x = \frac{1}{2}$. We restrict the control parameter r to the range $0 \leq r \leq 4$ so that (1) maps the interval $0 \leq x \leq 1$ into itself. (The behavior is much less interesting for other values of x and r—see Exercise 10.2.1.)

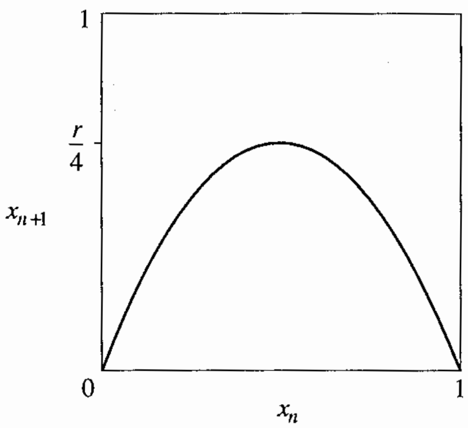

Figure 10.2.1

Period-Doubling

Suppose we fix r, choose some initial population x_0, and then use (1) to generate the subsequent x_n. What happens?

For small growth rate $r < 1$, the population always goes extinct: $x_n \to 0$ as $n \to \infty$. This gloomy result can be proven by cobwebbing (Exercise 10.2.2).

For $1 < r < 3$ the population grows and eventually reaches a nonzero steady state (Figure 10.2.2). The results are plotted here as a ***time series*** of x_n vs. n. To make the sequence clearer, we have connected the discrete points (n, x_n) by line segments, but remember that only the corners of the jagged curves are meaningful.

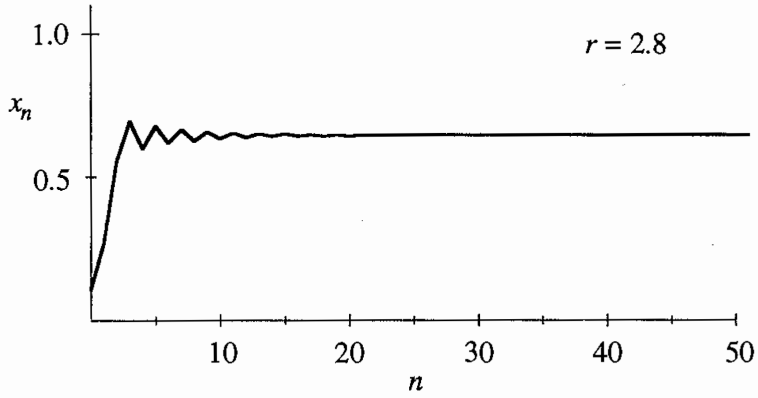

Figure 10.2.2

For larger r, say $r = 3.3$, the population builds up again but now *oscillates* about the former steady state, alternating between a large population in one generation and a smaller population in the next (Figure 10.2.3). This type of oscillation, in which x_n repeats every *two* iterations, is called a ***period-2 cycle***.

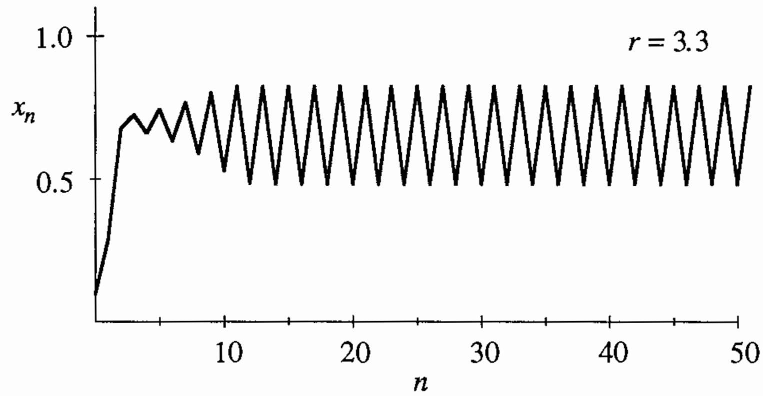

Figure 10.2.3

At still larger r, say $r = 3.5$, the population approaches a cycle that now repeats every *four* generations; the previous cycle has doubled its period to ***period-4*** (Figure 10.2.4).

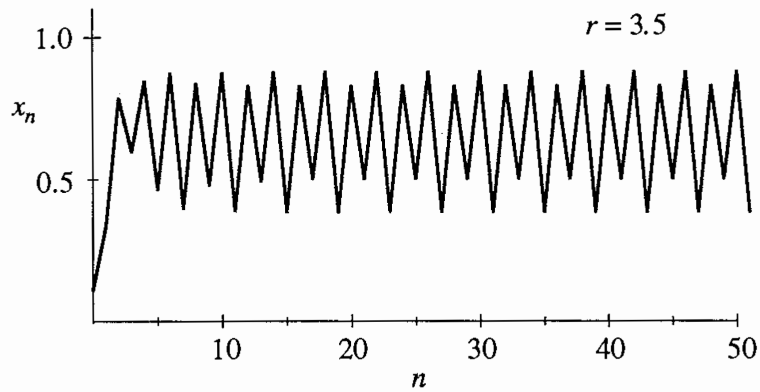

Figure 10.2.4

Further ***period-doublings*** to cycles of period 8, 16, 32, . . . , occur as r increases. Specifically, let r_n denote the value of r where a 2^n-cycle first appears. Then computer experiments reveal that

$r_1 = 3$	(period 2 is born)
$r_2 = 3.449...$	4
$r_3 = 3.54409...$	8
$r_4 = 3.5644...$	16
$r_5 = 3.568759...$	32
$\vdots$	$\vdots$
$r_\infty = 3.569946...$	∞

Note that the successive bifurcations come faster and faster. Ultimately the r_n converge to a limiting value r_∞. The convergence is essentially geometric: in the limit of large n, the distance between successive transitions shrinks by a constant factor

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669\ldots.$$

We'll have a lot more to say about this number in Section 10.6.

Chaos and Periodic Windows

According to Gleick (1987, p. 69), May wrote the logistic map on a corridor blackboard as a problem for his graduate students and asked, "*What the Christ happens for* $r > r_\infty$?" The answer turns out to be complicated: For many values of r, the sequence $\{x_n\}$ never settles down to a fixed point or a periodic orbit—instead the long-term behavior is aperiodic, as in Figure 10.2.5. This is a discrete-time version of the chaos we encountered earlier in our study of the Lorenz equations (Chapter 9).

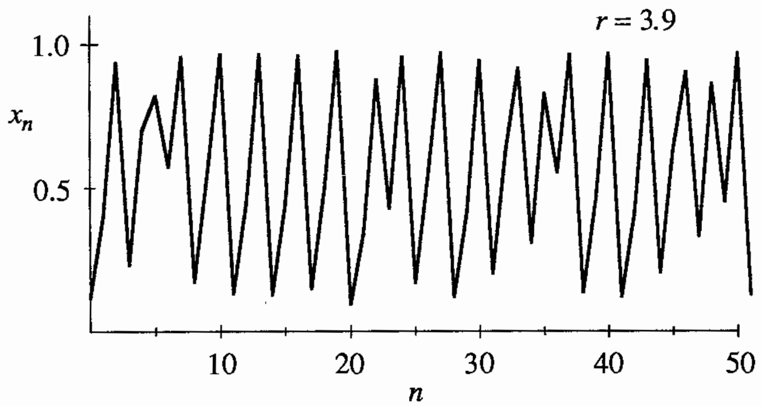

Figure 10.2.5

The corresponding cobweb diagram is impressively complex (Figure 10.2.6).

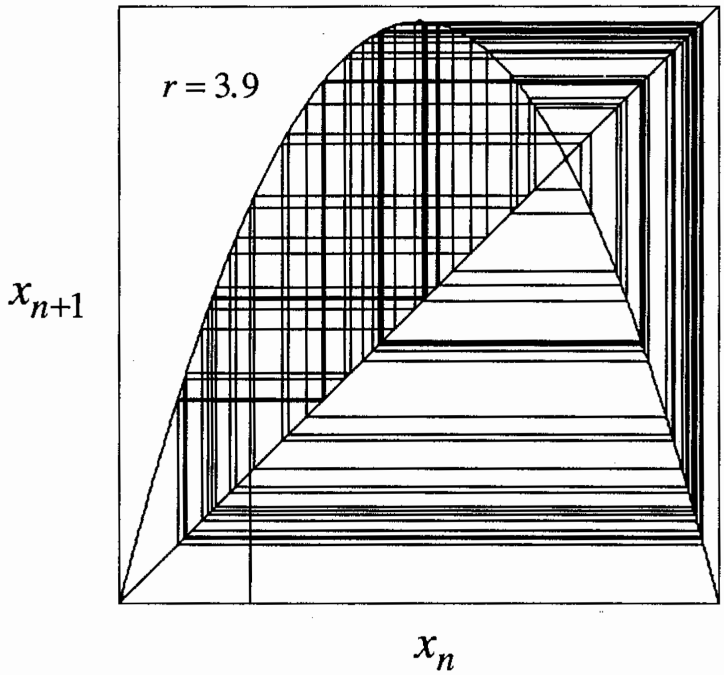

Figure 10.2.6

You might guess that the system would become more and more chaotic as r increases, but in fact the dynamics are more subtle than that. To see the long-term behavior for *all* values of r at once, we plot the ***orbit diagram***, a magnificent picture that has become an icon of nonlinear dynamics (Figure 10.2.7). Figure 10.2.7 plots the system's attractor as a function of r. To generate the orbit diagram for yourself, you'll need to write a computer program with two "loops." First, choose a value of r. Then generate an orbit starting from some random initial condition x_0. Iterate for 300 cycles or so, to allow the system to settle down to its eventual behavior. Once the transients have decayed, plot many points, say $x_{301}, \ldots, x_{600}$ above that r. Then move to an adjacent value of r and repeat, eventually sweeping across the whole picture.

Figure 10.2.7 shows the most interesting part of the diagram, in the region $3.4 \le r \le 4$. At $r = 3.4$, the attractor is a period-2 cycle, as indicated by the two branches. As r increases, both branches split simultaneously, yielding a period-4 cycle. This splitting is the period-doubling bifurcation mentioned earlier. A cascade of further period-doublings occurs as r increases, yielding period-8, period-16, and so on, until at $r = r_\infty \approx 3.57$, the map becomes chaotic and the attractor changes from a finite to an infinite set of points.

For $r > r_\infty$ the orbit diagram reveals an unexpected mixture of order and chaos, with ***periodic windows*** interspersed between chaotic clouds of dots. The large window beginning near $r \approx 3.83$ contains a stable period-3 cycle. A blow-up of part of the period-3 window is shown in the lower panel of Figure 10.2.7. Fantastically, a copy of the orbit diagram reappears in miniature!

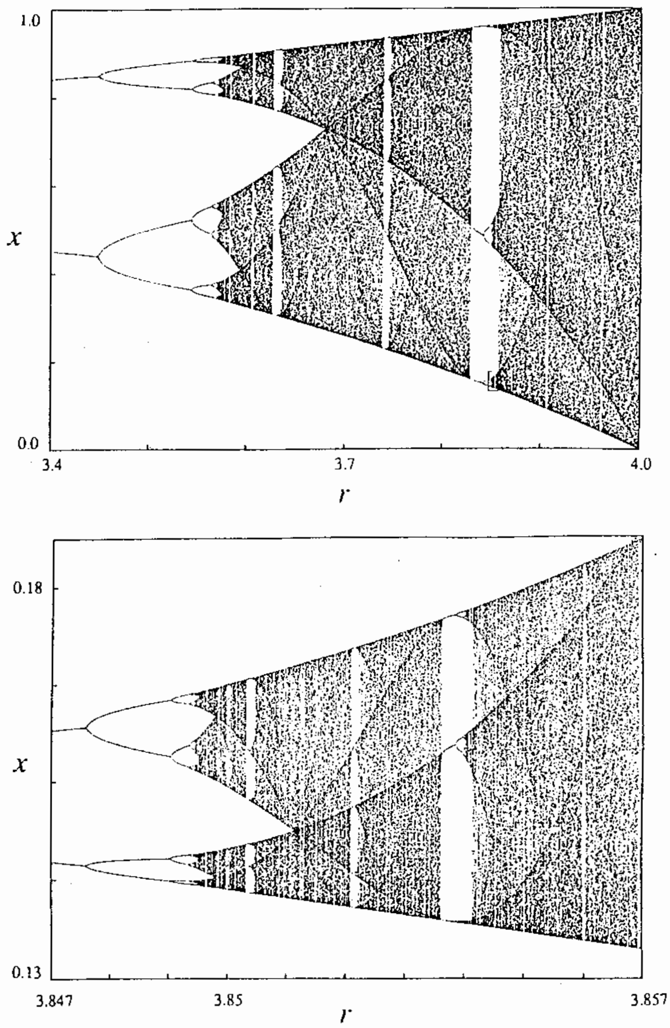

Figure 10.2.7 Campbell (1979), p. 35, courtesy of Roger Eckhardt

10.3 Logistic Map: Analysis

The numerical results of the last section raise many tantalizing questions. Let's try to answer a few of the more straightforward ones.

EXAMPLE 10.3.1:

Consider the logistic map $x_{n+1} = rx_n(1-x_n)$ for $0 \le x_n \le 1$ and $0 \le r \le 4$. Find all the fixed points and determine their stability.

Solution: The fixed points satisfy $x^* = f(x^*) = rx^*(1-x^*)$. Hence $x^* = 0$ or $1 = r(1-x^*)$, i.e., $x^* = 1 - \frac{1}{r}$. The origin is a fixed point for all r, whereas $x^* = 1 - \frac{1}{r}$ is in the range of allowable x only if $r \ge 1$.

Stability depends on the multiplier $f'(x^*) = r - 2rx^*$. Since $f'(0) = r$, the origin is stable for $r < 1$ and unstable for $r > 1$. At the other fixed point,