Logistic Map

1976

Bob May

$\dagger$ 2020

(Robert)

X

$n = 0, 1, 2, \ldots\ldots$

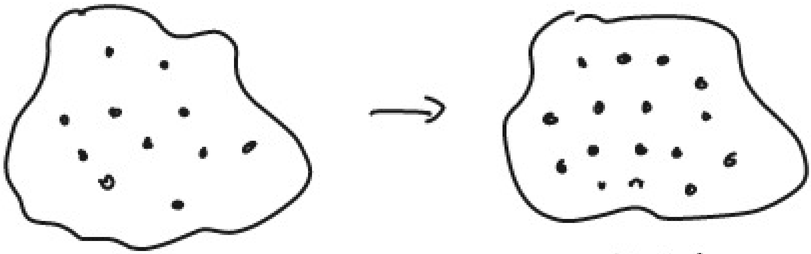
$x_n \qquad x_0, x_1, x_2, x_3 \ldots\ldots$

$$\boxed{x_{n+1} = f(x_n) \qquad \text{1-d map}}$$

$$x_{n+1} = \lambda x_n (1 - x_n)$$

$0 \leq X \leq 1 \quad 0 < \lambda < 4$

$n = 0, 1, 2, \ldots$



n $\qquad\qquad$ $n+1$

U_n $\qquad\qquad$ U_{n+1}

$$\boxed{U_{n+1} = \lambda U_n}$$

reproduction rate

$\lambda \geq$

$$U_n = \lambda^n u_0$$

$$= e^{n \cdot \ln \lambda} u_0$$

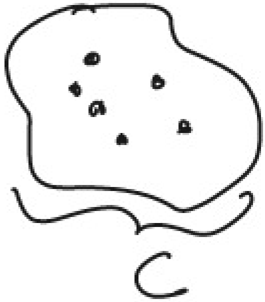
$\lambda > 1$

$\lambda < 1$

$$\boxed{u_{n+1} = \lambda u_n}$$

λ

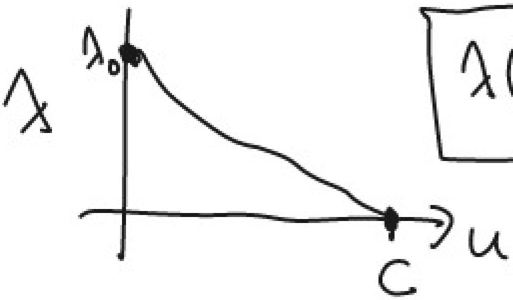
$\lambda(u)$



$u \ll C$

$\lambda(u) \approx \lambda_0$

$u \approx C \quad \lambda(u) \to 0$



$$\boxed{\lambda(u) = \lambda_0\left(1 - \frac{u}{C}\right)}$$

$$\frac{u_{n+1}}{C} = \lambda_0\left(1 - \frac{u_n}{C}\right)\frac{u_n}{C}$$

$X_n = u_n / C$

$$\boxed{X_{n+1} = \lambda (1 - X_n) X_n}$$

λ fixed $\quad x_0$

$x_0 \to x_1 \to x_2 \to x_3 \dots\dots$

$X_{n+1} = f(x_n)$ $\quad \boxed{f(x) = \lambda x (1 - x)}$

$x_0 = x^* \quad x_1 = x^* \quad x_2 = x^*$

$$\boxed{x^* = f(x^*)}$$

$$x^* = \lambda x^* (1 - x^*)$$

1.) $x^* = 0$ Fixpoint

2.) $1 = \lambda(1 - x^*) = \lambda - \lambda x^*$

$$\Rightarrow \quad \boxed{x^* = 1 - \frac{1}{\lambda}}$$

$0 \leq \lambda \leq 4$

$0 \leq x$

$\boxed{\text{only if } \lambda > 1!}$

e.g. $\lambda = 2 \quad x^* = 1/2$

$$\boxed{\begin{array}{l} x_{n+1} = f(x_n) \\ x^* \Rightarrow \quad x^* = f(x^*) \end{array}}$$

$$x_0 = x^* + \delta x_0$$ perturbation very small

$$\delta x_0 = x_0 - x^*$$

$$x_1 = f(x_0) = f(x^* + \delta x_0)$$

$$\approx f(x^*) + \delta x_0 f'(x$$

$$x_1 = x^* + \delta x_0 f'(x^*)$$

$$\underbrace{x_1 - x^*} = \delta x_0 f'(x^*)$$

$$\boxed{\delta x_1 = f'(x^*) \delta x_0}$$

$$|f'(x^*)| > 1 \qquad |\delta x_1| > |\delta x_0|$$

$\Rightarrow$ unstable

$$|f'(x^*)| < 1 \qquad |\delta x_1| < |\delta x_0|$$

$\Rightarrow$ stable

$$x_{n+1} = \lambda x_n (1 - x_n)$$

$x^* = 0$

$x^* = 1 - \frac{1}{\lambda}$

$$\boxed{f(x) = \lambda x (1 - x)}$$

$$f(x) = \lambda x - \lambda x^2$$

$$f'(x) = \lambda - 2\lambda x$$

$x^* = 0$ $\quad f'(x^*) = f'(0) = \lambda$

stable if $\lambda < 1$

unstable if $\lambda > 1$

$x^* = 1 - \frac{1}{\lambda}$

$$f'\left(1 - \frac{1}{\lambda}\right) = \lambda - 2\lambda\left(1 - \frac{1}{\lambda}\right)$$

$$= \lambda - 2\lambda + 2$$

$$= 2 - \lambda$$

$|f'(x^*)| = |2-\lambda|$ $\quad$ $\lambda >$

if $1 < \lambda < 3$ $\quad$ $|2-\lambda| < 1$

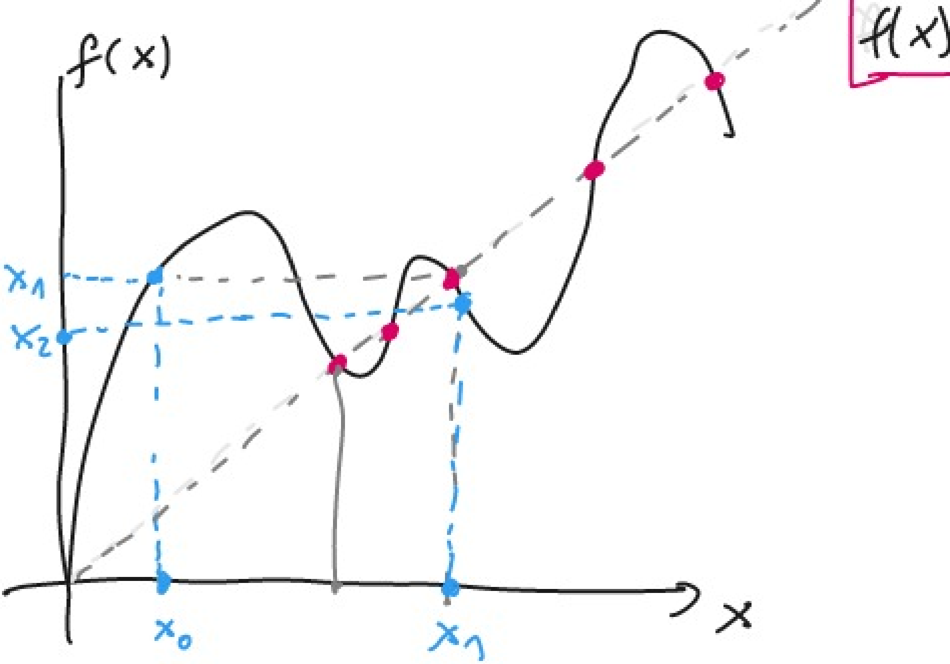
$\Rightarrow x^*$ is stable

if $\lambda > 3$ $\quad$ $|2-\lambda| > 1$

$\Rightarrow x^* \Rightarrow$ unstable

$\boxed{\lambda > 3}$ $\quad$ $x^* = 0$ $\quad$ $x^* = 1 - \frac{1}{\lambda}$

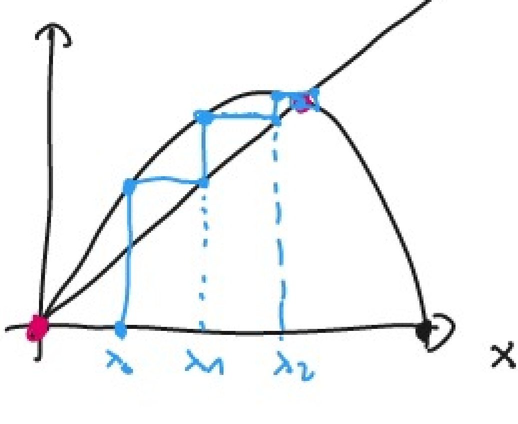
both unstable

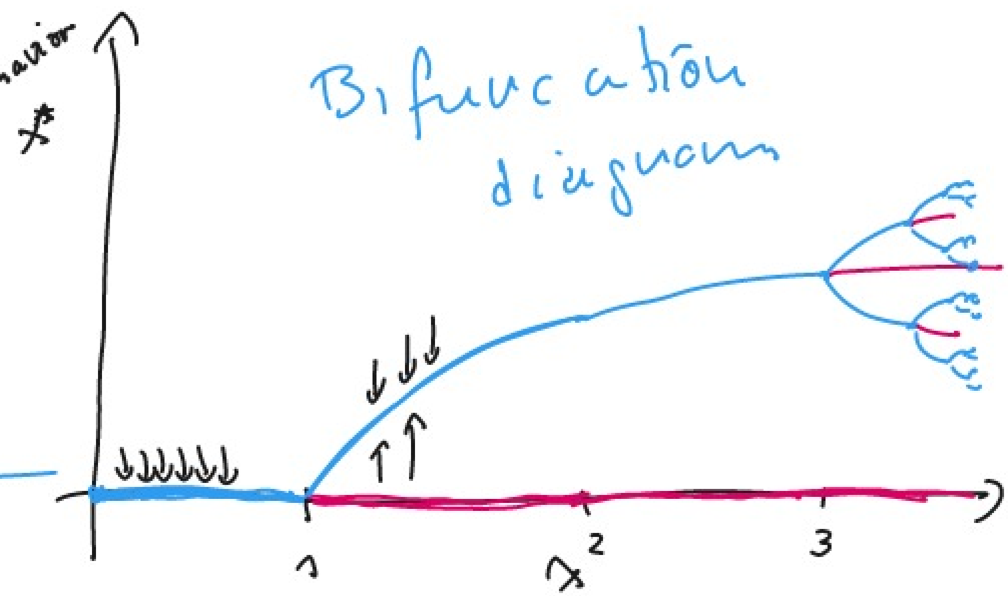


$$x_{n+1} = \lambda x_n (1 - x_n)$$

$$\boxed{x_{n+1} = f(x_n)} \qquad x^* = f(x^*)$$

$\lambda = 2$





$x^* = 1 - \frac{1}{\lambda}$